A SUPER-FLAG LANDAU MODEL

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We consider the quantum mechanics of a particle on the coset superspace $SU(2|1)/[U(1)\times U(1)]$, which is a super-flag manifold with $SU(2)/U(1) \cong S^2$ 'body'. By incorporating the Wess-Zumino terms associated with the $U(1) \times U(1)$ stability group, we obtain an exactly solvable super-generalization of the Landau model for a charged particle on the sphere. We solve this model using the factorization method. Remarkably, the physical Hilbert space is finite-dimensional because the number of admissible Landau levels is bounded by a combination of the $U(1)$ charges. The level saturating the bound has a wavefunction in a shortened, degenerate, irrep of $SU(2|1)$.

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1. Introduction

In 1930 Landau posed and solved the problem of a quantum particle in a plane orthogonal to a uniform magnetic field, showing in particular that the particle's energy is restricted to a series of 'Landau levels' [1]. It is now customary to call a 'Landau model' any problem in which a quantum particle is confined to a surface orthogonal to a magnetic field that is uniform on the surface. A case in point is the Landau model of a particle on a unit sphere in $\mathbb{E}^3$ with a magnetic monopole at the center. This model was introduced by Haldane in the context of the Quantum Hall Effect [2], and has many fascinating features. For example, it is exactly soluble [3]. When restricted to the lowest Landau level (LLL) the sphere becomes the phase space rather than the configuration space, and this leads to a physical realization of the fuzzy sphere [4].

Ian Kogan worked on aspects of Landau models [5] around the same time that he developed the idea of the 'monopole bag' [6] in which a closed axion domain wall is supported against collapse by the electric charge induced on it by a magnetic monopole inside. Perhaps he saw a connection? The 'monopole bag' was what inspired one of us to observe that a closed D2-brane carrying a net electric charge would appear to be a D0-brane [7], and it is now appreciated that there are circumstances in which it is energetically favorable for D0-branes to 'expand' into a fuzzy spherical D2-brane [8, 9]. The fuzzy sphere thus appears as a common theme.

Recently, we showed how the fuzzy *supersphere* emerges from the LLL quantum mechanics of a particle on the coset superspace $SU(2|1)/U(1|1)$ [10]. There is a natural extension of this model to a full Landau model but this involves terms quadratic in time-derivatives of the Grassmann odd variables, and such terms would normally be considered 'higher-derivative'. This is one of the reasons that supergroups such as $SU(2|1)$ do not normally appear as symmetry groups in physical problems.

Here we show that 'higher-derivative' fermion terms *can* be avoided in an $SU(2|1)$-invariant extension of the full Landau problem for a particle on the sphere, but instead of the supersphere one has to consider the coset superspace

$$SU(2|1)/[U(1) \times U(1)] \equiv SF\,. \tag{1.1}$$

This again has $SU(2)/U(1) \cong S^2$ 'body' and is a homogeneous Kähler superspace, but it is not a symmetric superspace. It is a flag supermanifold, analogous to the flag manifold $SU(3)/[U(1) \times U(1)]$. For the sake of brevity, we call it the 'super-flag' (SF). This super-extension of the sphere allows the construction of a Landau-type model with a ‘canonical’ fermion kinetic term arising from Wess-Zumino (WZ) terms associated with the two $U(1)$ factors of the stability subgroup. The phase space of this model has real dimension (4|4), so the configuration space has real dimension (2|2) with S^2 body, exactly as one would have for a particle on the supersphere, but without the ‘higher-derivative’ fermion kinetic term.

We quantize this model using techniques explained recently in [10, 11]: this leads to a Hilbert space spanned by ‘chiral’ superfields on SF. The Hamiltonian is shown to act in this physical subspace and we use Schroedinger's factorization method [12] to determine its eigenstates and eigenvalues, following the application of this method to the Landau model for a particle on the sphere [13]. Remarkably, we find that the number of Landau levels is finite, in contrast to the infinite number of levels in the bosonic case. This is because wavefunctions with positive norm exist only for $\ell \leq 2M$, where ℓ is the number of the Landau level and M is the properly normalized positive eigenvalue of some combination of two $U(1)$ charges. The full Hilbert space is therefore finite dimensional!

2. Super-flag geometry

The supergroup $SU(2|1)$ can be defined as the group of $(1|2) \times (1|2)$ unitary supermatrices of unit super-determinant. A parametrization of $SU(2|1)$ that makes manifest the Kähler property of its coset superspace $SU(2|1)/[U(1)\times U(1)]$ can be found following steps analogous to those spelled out for $SU(3)/[U(1)\times U(1)]$ in [14]. The group $SU(2|1)$ acts linearly on vectors in a vector superspace of dimension (1|2). A simple choice of basis in this superspace is provided by the columns of the supermatrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -\xi^2 & 1 & 0 \\ -\xi^1 & z & 1 \end{pmatrix} , \tag{2.1}$$

where z is a complex variable and ξ^i $(i = 1, 2)$ are complex anticommuting variables, with complex conjugates $\bar{\xi}_i$. By an application of the Gramm-Schmidt procedure we can transform the above supermatrix into a unitary supermatrix U for which the three column supervectors are orthonormal. This ensures that $U \in SU(2|1)$. One finds that

$$U = \left(\frac{1}{K_1^{\frac{1}{2}}} \begin{bmatrix} 1 \\ -\xi^2 \\ -\xi^1 \end{bmatrix} \; \left(\frac{K_1}{K_2}\right)^{\frac{1}{2}} \begin{bmatrix} \left(\bar{\xi}_2 + z\bar{\xi}_1\right)/{K_1}^2 \\ 1 - \bar{\xi}_1 \left(\xi^1 - z\xi^2\right) \\ z + \bar{\xi}_2 \left(\xi^1 - z\xi^2\right) \end{bmatrix} \; \frac{1}{K_2^{\frac{1}{2}}} \begin{bmatrix} \bar{\xi}_1 - \bar{z}\bar{\xi}_2 \\ -\bar{z} \\ 1 \end{bmatrix} \right) , \tag{2.2}$$

where

$$K_1 = 1 + \bar{\xi}_1\xi^1 + \bar{\xi}_2\xi^2\,, \qquad K_2 = 1 + \bar{z}z + \left(\xi^1 - z\xi^2\right)\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right). \tag{2.3}$$

The general $SU(2|1)$ supermatrix can be written in the form Uh, where h is a diagonal unitary supermatrix with unit superdeterminant parametrized by two angles. This means that the unitary supermatrix U provides a parametrization of the coset superspace $SU(2|1)/[U(1) \times U(1)]$.

To compute the Cartan forms and $U(1)$ connections for $SU(2|1)/[U(1) \times U(1)]$, we write the Lie superalgebra valued 1-form $U^{-1}dU$ as

$$U^{-1}dU \equiv \Omega = \begin{pmatrix} 0 & \bar{E}_2 & \bar{E}_1 \\ -E^2 & 0 & -\bar{E}_+ \\ -E^1 & E^+ & 0 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} \mathcal{B} & 0 & 0 \\ 0 & \mathcal{B}-\mathcal{A} & 0 \\ 0 & 0 & \mathcal{A} \end{pmatrix}. \tag{2.4}$$

The Cartan 1-forms are $E^A = (E^+, E^1, E^2)$ and their complex conjugates are $\bar{E}_A = (\bar{E}_+, \bar{E}_1, \bar{E}_2)$. One finds that

$$E^A = dZ^M E_M{}^A\,, \qquad \bar{E}_A = d\bar{Z}_M \bar{E}^M{}_A\,, \tag{2.5}$$

where $Z^M = (z, \xi^1, \xi^2)$ are the complex coordinates and $\bar{Z}_M = (\bar{z}, \bar{\xi}_1, \bar{\xi}_2)$ their complex conjugates; this defines the (complex) supervielbein $E_M{}^A$. Using the inverse supervielbein $E_A{}^M$, and its complex conjugate $\bar{E}^A{}_M$, we define the complex supercovariant derivative D_A and its complex conjugate $\bar{D}^A$ as

$$D_A = E_A{}^M\partial_M\,, \qquad \bar{D}^A = \bar{E}^A{}_M\bar{\partial}^M\,. \tag{2.6}$$

A computation shows that

$$\begin{aligned}
E^+ &= K_1^{-\frac{1}{2}}K_2^{-1}\left[dz - K_1^{-1}\left(d\xi^1 - zd\xi^2\right)\left(\bar{\xi}_2 + z\bar{\xi}_1\right)\right],\\
E^1 &= (K_1K_2)^{-\frac{1}{2}}\left[d\xi^1 - zd\xi^2\right],\\
E^2 &= K_2^{-\frac{1}{2}}\left[d\xi^1\left(\bar{z} - \xi^2\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\right) + d\xi^2\left(1 + \xi^1\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\right)\right]
\end{aligned} \tag{2.7}$$

and that

$$\begin{aligned}
D_+ &= K_1^{\frac{1}{2}}K_2\partial_z\,,\\
D_1 &= K_2^{\frac{1}{2}}K_1^{-\frac{1}{2}}\left(\bar{\xi}_2 + z\bar{\xi}_1\right)\partial_z\\
&\quad + K_1^{\frac{1}{2}}K_2^{-\frac{1}{2}}\left\{\left[1 + \xi^1\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\right]\partial_{\xi^1} - \left[\bar{z} - \xi^2\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\right]\partial_{\xi^2}\right\},\\
D_2 &= K_2^{-\frac{1}{2}}\left(z\partial_{\xi^1} + \partial_{\xi^2}\right).
\end{aligned} \tag{2.8}$$

For the $U(1)$ connections $\mathcal{A}$ and $\mathcal{B}$ we have, similarly, that

$$\mathcal{A} = dZ^M \mathcal{A}_M + c.c.\,, \qquad \mathcal{B} = dZ^M \mathcal{B}_M + c.c. \tag{2.9}$$

and a calculation shows that

$$\mathcal{A} = -idZ^M \partial_M \log K_2 + c.c.\,, \qquad \mathcal{B} = idZ^M \partial_M \log K_1 + c.c.\,. \tag{2.10}$$

The $SU(2|1)$ transformations of the superspace coordinates $Z^M, \bar{Z}_M$ can be found as follows. Let us write $U(\mathcal{Z})$ for the unitary supermatrix (2.2) where

$$\mathcal{Z} = (Z^M, \bar{Z}_M)\,. \tag{2.11}$$

For any element $U \in SU(2|1)$ we have

$$UU(\mathcal{Z}) = U(\mathcal{Z}')h \tag{2.12}$$

for some diagonal unitary matrix h in the $U(1) \times U(1)$ stability subgroup. We choose h to have the expansion

$$h = I + \left(\alpha \tilde{J}_3 + \beta \tilde{B}\right) + \cdots\,, \tag{2.13}$$

where

$$\tilde{J}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

If one now chooses $U = U(\Delta, \bar{\Delta})$ for constant infinitesimal parameter $\Delta = (a, \epsilon^1, \epsilon^2)$, where a is Grassmann-even and ϵ^i $(i = 1, 2)$ Grassmann-odd, then one finds that $Z' = Z + \delta Z$, where

$$\begin{aligned} \delta z &= a + \bar{a} z^2 - (\bar{\epsilon}_2 + z\bar{\epsilon}_1)\left(\xi^1 - z\xi^2\right), \\ \delta\xi^1 &= a\xi^2 + \epsilon^1 + (\bar{\epsilon}\cdot\xi)\,\xi^1\,, \\ \delta\xi^2 &= -\bar{a}\xi^1 + \epsilon^2 + (\bar{\epsilon}\cdot\xi)\,\xi^2 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \alpha(\mathcal{Z}, \Delta, \bar{\Delta}) &= \frac{1}{2}\left[\bar{a}z - a\bar{z} + (\bar{\xi}_1 - \bar{z}\bar{\xi}_2)\epsilon^1 - \bar{\epsilon}_1(\xi^1 - z\xi^2)\right], \\ \beta(\mathcal{Z}, \Delta, \bar{\Delta}) &= \frac{1}{2}\left(\bar{\xi}\cdot\epsilon - \bar{\epsilon}\cdot\xi\right)\,. \end{aligned} \tag{2.16}$$

The $U(1) \times U(1)$ transformations of the coordinates corresponding to $\mathcal{Z}$-independent parameters α_0 and β_0 in (2.13) $(\bar{\alpha}_0 = -\alpha_0, \bar{\beta}_0 = -\beta_0)$ are as follows

$$\delta z = (2\alpha_0 - \beta_0)\, z\,, \quad \delta\xi^1 = (\alpha_0 - \beta_0)\, \xi^1\,, \qquad \delta\xi^2 = -\alpha_0\, \xi^2\,. \tag{2.17}$$

We have therefore shown that, in the chosen parametrization of the superflag, the $SU(2|1)$ transformations of $(z, \bar{z}, \xi^i, \bar{\xi}_i)$ are analytic: the coordinates $Z = (z, \xi^i)$ transform among themselves, and the same is true for $\bar{Z} = (\bar{z}, \bar{\xi}_i)$. Various other $SU(2|1)$ invariant subspaces determine the various types of superfields that one can define on the superflag, as we now explain.

3. Super-flag superfields

In accord with the general procedure of nonlinear realizations, superfields given on SF are characterized by two external $U(1)$ charges. The corresponding operators $\hat{J}_3$ and $\hat{B}$ are the 'matrix' parts of the differential operators representing the $U(1)\times U(1)$ subgroup of $SU(2|1)$ (in other words, $\hat{J}_3$ and $\hat{B}$ count external $U(1)$ charges of the superfield). The only superfields that we need to consider are those that are eigenfunctions of $\hat{J}_3$ and $\hat{B}_3$ with eigenvalues $2N$ and $2M$, respectively:

$$\hat{J}_3\, \Psi^{(N,M)}(\mathcal{Z}) = 2N\, \Psi^{(N,M)}(\mathcal{Z})\,, \quad \hat{B}\, \Psi^{(N,M)}(\mathcal{Z}) = 2M\, \Psi^{(N,M)}(\mathcal{Z})\,. \tag{3.1}$$

Such superfields transform as

$$\Psi^{(N,M)\,\prime}(\mathcal{Z}') = h(\mathcal{Z}, \Delta, \bar{\Delta})\Psi^{(N,M)}(\mathcal{Z})\,. \tag{3.2}$$

In infinitesimal form,

$$\delta\Psi^{(N,M)}(\mathcal{Z}) = 2\left[N\,\alpha(\mathcal{Z}, \Delta, \bar{\Delta}) + M\,\beta(\mathcal{Z}, \Delta, \bar{\Delta})\right]\Psi^{(N,M)}(\mathcal{Z})\,. \tag{3.3}$$

The $U(1)\times U(1)$ gauge covariant differential of a general superfield Ψ on SF is

$$\mathcal{D}\Psi = \left(d - \frac{i}{2}\mathcal{A}\hat{J}_3 - \frac{i}{2}\mathcal{B}\hat{B}\right)\Psi = \left(E^A\mathcal{D}_A + \bar{E}_A\bar{\mathcal{D}}^A\right)\Psi\,, \tag{3.4}$$

which defines the gauge covariant derivatives $\mathcal{D}_A$. Using the identities

$$\begin{aligned} &D_1 K_2 = K_2^{\frac{3}{2}} K_1^{-\frac{1}{2}} \bar{\xi}_1\,, \quad D_1 K_1 = -K_1^{\frac{3}{2}} K_2^{-\frac{1}{2}} (\bar{\xi}_1 - \bar{z}\bar{\xi}_2)\,, \\ &D_2 K_2 = 0\,, \qquad D_2 K_1 = -K_2^{-\frac{1}{2}} (\bar{\xi}_2 + z\bar{\xi}_1)\,, \end{aligned} \tag{3.5}$$

one finds that

$$\begin{aligned}
&\mathcal{D}_+ = D_+ - \frac{1}{2}K_1^{\frac{1}{2}}\partial_z K_2\,\hat{J}_3\,,\ \ \mathcal{D}^+ = \overline{\mathcal{D}_+} = D^+ + \frac{1}{2}K_1^{\frac{1}{2}}\partial_{\bar z} K_2\,\hat{J}_3\,,\\
&\mathcal{D}_1 = D_1 - \frac{1}{2}K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}\bar\xi_1\,\hat{J}_3 - \frac{1}{2}K_1^{\frac{1}{2}}K_2^{-\frac{1}{2}}(\bar\xi_1 - \bar z\bar\xi_2)\,\hat{B}\,,\\
&\bar{\mathcal{D}}^1 = \bar{D}^1 + \frac{1}{2}K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}\xi^1\,\hat{J}_3 + \frac{1}{2}K_1^{\frac{1}{2}}K_2^{-\frac{1}{2}}(\xi^1 - z\xi^2)\,\hat{B}\,,\\
&\mathcal{D}_2 = D_2 - \frac{1}{2}K_1^{-1}K_2^{-\frac{1}{2}}(\bar\xi_2 + z\bar\xi_1)\,\hat{B}\,,\\
&\bar{\mathcal{D}}^2 = \bar{D}^2 + \frac{1}{2}K_1^{-1}K_2^{-\frac{1}{2}}(\xi^2 + \bar z\xi^1)\,\hat{B}\,.
\end{aligned} \tag{3.6}$$

The geometry of the coset superspace $SU(2|1)/[U(1)\times U(1)]$ is now encoded in the (anti)commutation relations

$$\left[\mathcal{D}_+,\mathcal{D}^+\right] = \hat{J}_3\,, \tag{3.7}$$

$$\{\mathcal{D}_1,\mathcal{D}_1\} = \{\mathcal{D}_2,\mathcal{D}_2\} = \{\mathcal{D}_1,\mathcal{D}_2\} = 0 \quad \text{and c.c.}\,, \tag{3.8}$$

$$\left\{\mathcal{D}_1,\bar{\mathcal{D}}^1\right\} = (\hat{J}_3 + \hat{B})\,,\quad \left\{\mathcal{D}_2,\bar{\mathcal{D}}^2\right\} = \hat{B}\,, \tag{3.9}$$

$$\left\{\mathcal{D}_1,\bar{\mathcal{D}}^2\right\} = -\mathcal{D}_+\,,\quad \left\{\mathcal{D}_2,\bar{\mathcal{D}}^1\right\} = \mathcal{D}^+\,, \tag{3.10}$$

$$\begin{aligned}
&\left[\mathcal{D}_+,\bar{\mathcal{D}}^1\right] = -\bar{\mathcal{D}}^2\,,\quad \left[\mathcal{D}_+,\bar{\mathcal{D}}^2\right] = 0\,,\\
&\left[\mathcal{D}^+,\bar{\mathcal{D}}^2\right] = \bar{\mathcal{D}}^1\,,\quad \left[\mathcal{D}^+,\bar{\mathcal{D}}^1\right] = 0\,,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&[\mathcal{D}_+,\mathcal{D}_1] = 0\,,\quad \left[\mathcal{D}_+,\bar{\mathcal{D}}_2\right] = \mathcal{D}_1\,,\\
&\left[\mathcal{D}^+,\mathcal{D}_2\right] = 0\,,\quad \left[\mathcal{D}^+,\mathcal{D}^1\right] = -\mathcal{D}_2\,.
\end{aligned} \tag{3.12}$$

Using the fact that the charges of the covariant derivatives are opposite to those of the Cartan forms, the $U(1)\times U(1)$ assignments of both can be worked out from the transformation rule

$$\Omega\,' = h\Omega h^{-1} - d\alpha\tilde{J}_3 - d\beta\tilde{B}\,. \tag{3.13}$$

Here we record the result for the $U(1)$ charges of the covariant derivatives:

$$\hat{J}_3\,\mathcal{D}_+ = -2\mathcal{D}_+\,,\ \ \hat{J}_3\,\mathcal{D}_1 = -\mathcal{D}_1\,,\ \ \hat{J}_3\,\mathcal{D}_2 = \mathcal{D}_2\,, \tag{3.14}$$

$$\hat{B}\,\mathcal{D}_+ = \mathcal{D}_+\,,\ \ \hat{B}\,\mathcal{D}_1 = \mathcal{D}_1\,,\ \ \hat{B}\,\mathcal{D}_2 = 0\,. \tag{3.15}$$

Note that, instead of $\hat{B}$, it is sometimes more convenient to use the combination [a]

$$\hat{F} = 2\hat{B} + \hat{J}_3\,, \tag{3.16}$$

[a] This is just the matrix part of the $U(1)$ generator $J_3 + 2B$ that commutes with the $SU(2)$ generators.

which is distinguished by the fact that the S^2 covariant derivatives $\mathcal{D}_+, \mathcal{D}^+(= \bar{\mathcal{D}}_+)$ (and the corresponding Cartan forms) have $\hat{F}$ charge zero, while both spinor derivatives have $\hat{F}$ charge 1,

$$\hat{F}\,\mathcal{D}_2 = \mathcal{D}_2\,, \quad \hat{F}\,\mathcal{D}_1 = \mathcal{D}_1\,. \tag{3.17}$$

It will be convenient to set

$$\mathcal{D}_+ = K_1^{\frac{1}{2}} K_2 \nabla_z^{(N)}\,, \quad \mathcal{D}^+ = K_1^{\frac{1}{2}} K_2 \nabla_{\bar{z}}^{(N)}\,, \tag{3.18}$$

which defines the 'semi-covariant' derivatives

$$\begin{aligned}\nabla_z^{(N)} &= \partial_z - iN\mathcal{A}_z = \partial_z - N\partial_z \log K_2\,,\\ \nabla_{\bar{z}}^{(N)} &= \partial_{\bar{z}} - iN\mathcal{A}_{\bar{z}} = \partial_{\bar{z}} + N\partial_{\bar{z}} \log K_2\,.\end{aligned} \tag{3.19}$$

The N dependence arises here because we assume that the covariant derivatives act on superfields $\Psi^{(N,M)}$ obeying (3.1). It is easy to check that (3.7) is equivalent to the following commutation relation between the 'semi-covariant' derivatives

$$\left[\nabla_z^{(N)}, \nabla_{\bar{z}}^{(N)}\right] = 2K_1^{-1}K_2^{-2}\,N\,. \tag{3.20}$$

This can also be checked by using the identity

$$\partial_z\partial_{\bar{z}} \log K_2 = K_1^{-1}K_2^{-2}\,.$$

Let us now note a few important corollaries of the (anti)commutation relations:

- For any value of N and M it is consistent to consider covariantly chiral or anti-chiral superfields [b]

$$\text{either (a)} \;\; \bar{\mathcal{D}}^i \Psi^{(N,M)} = 0 \quad \text{or (b)} \;\; \mathcal{D}_i \tilde{\Psi}^{N,M} = 0\,. \tag{3.21}$$

- Equations (3.11) imply that the S^2 covariant derivatives $\mathcal{D}^+, \mathcal{D}^+$ form a closed subset with $\bar{\mathcal{D}}^i$ or $\mathcal{D}_i$ and so preserve chirality. In other words, they yield some chiral (anti-chiral) superfield when acting on $\Psi^{(N,M)}$ or $\tilde{\Psi}^{(N,M)}$, as defined in (3.21). Since these derivatives carry non-zero $U(1)$ charges, the charges are shifted from (N,M) to $(N-1, M+1/2)$ for $\mathcal{D}_+\Psi^{(N,M)}$ and from (N,M) to $(N+1, M-1/2)$

[b] As an aside, let us note that, besides the chirality conditions (3.21), one can consistently impose on a general $SU(2|1)$ superfield the Grassmann analyticity conditions $\mathcal{D}_2\Psi = \bar{\mathcal{D}}^1\Psi = \mathcal{D}^+\Psi = 0$ (or their complex conjugates). The covariant derivatives here form a set that is closed under (anti)commutation, as required for consistency of the conditions, which are analogs of the harmonic analyticity conditions in $N = 2, 4D$ supersymmetry [15].

for $\mathcal{D}^+\Psi^{(N,M)}$. In what follows we restrict our attention to the chiral superfields.

- One can consistently require chiral superfields to be covariantly holomorphic:

$$\text{either (a)}\ \ \mathcal{D}_+\Psi^{(N,M)} = 0 \quad \text{or (b)}\ \ \mathcal{D}^+\Psi^{(N,M)} = 0\,. \tag{3.22}$$

  However, a chiral superfield satisfying condition (a) is zero if $N > 0$ and one satisfying condition (b) is zero if $N < 0$. For chiral superfields with $N = 0$ one can impose *both* conditions (3.22), thus fully suppressing their $z, \bar{z}$ dependence.

- Equations (3.9), (3.10) imply that for $M = 0$ or $M = -N$ the covariant derivatives $\mathcal{D}_2$ and $\mathcal{D}^+$ or $\mathcal{D}_1$ and $\mathcal{D}_+$ together with $\bar{\mathcal{D}}^i$ form a set that is closed under (anti)commutation. Hence the chiral superfields with $M = 0$ or $M = -N$ can be subjected to the more stringent set of constraints

$$\mathcal{D}_2\Psi^{(N,0)} = \bar{\mathcal{D}}^i\Psi^{(N,0)} = 0\,, \quad \mathcal{D}^+\Psi^{(N,0)} = 0\,. \tag{3.23}$$

  Alternatively, one can impose the constraints

$$\mathcal{D}_1\Psi^{(N,-N)} = \bar{\mathcal{D}}^i\Psi^{(N,-N)} = 0\,, \quad \mathcal{D}_+\Psi^{(N,-N)} = 0\,. \tag{3.24}$$

  Thus chiral superfields can be made 'covariantly independent' of one more Grassmann coordinate, provided they are simultaneously assumed to be holomorphic or antiholomorphic, for $N \geq 0$ or $N \leq 0$, respectively. In what follows we shall deal with $N \geq 0$, thus specializing to the case (3.23).

For every set of conditions that may be imposed consistently on a superfield there is a corresponding invariant subset of the original coordinate set

$$\mathcal{Z} = (z, \bar{z}, \xi^1, \xi^2, \bar{\xi}_1, \bar{\xi}_2)\,. \tag{3.25}$$

As already mentioned, (z, ξ^i) is one such invariant subset, but there are others. For example, consider the new non-self-conjugate 'chiral' parametrization of SF

$$\tilde{\mathcal{Z}} = (z, \bar{z}_{sh}, \xi^1, \xi^2, \bar{\xi}_1, \bar{\xi}_2)\,, \tag{3.26}$$

where

$$\bar{z}_{sh} = \bar{z} - (\xi^2 + \bar{z}\xi^1)(\bar{\xi}_1 - \bar{z}\bar{\xi}_2)\,. \tag{3.27}$$

One can check that

$$\delta\bar{z}_{sh} = \bar{a} + a\bar{z}_{sh}^2 + (\bar{\epsilon}_1 - \bar{z}_{sh}\bar{\epsilon}_2)(\xi^2 + \bar{z}_{sh}\xi^1)\,, \tag{3.28}$$

so the 'chiral' subspace

$$\zeta_L = (z, \bar{z}_{sh}, \xi^i) \tag{3.29}$$

is closed under the action of $SU(2|1)$.

To see that the $SU(2|1)$ invariance of the chiral subspace of superspace is related to the existence of chiral superfields, we set

$$\Psi^{(N,M)} = K_1^M K_2^{-N}\Phi^{(N,M)}, \tag{3.30}$$

and observe that

$$\begin{aligned}
\bar{\mathcal{D}}^1\Psi^{(N,M)} &= K_1^{M-1/2}K_2^{-N-1/2}\bar{\nabla}^1\Phi^{(N,M)}\,,\\
\bar{\mathcal{D}}^2\Psi^{(N,M)} &= K_1^{M}K_2^{-N+1/2}\bar{\nabla}^2\Phi^{(N,M)}\,,\\
\mathcal{D}^+\Psi^{(N,M)} &= K_1^{M-1/2}K_2^{-(N+1)}\nabla_{\bar{z}}^{(N)ch}\Phi^{(N,M)}
\end{aligned} \tag{3.31}$$

where

$$\bar{\nabla}^1 = K_1K_2\bar{D}^1\,, \quad \bar{\nabla}^2 = K_2^{-\frac{1}{2}}\bar{D}^2\,, \quad \nabla_{\bar{z}}^{(N)ch} = K_1^{\frac{1}{2}}K_2D_- = K_1K_2^2\partial_{\bar{z}}\,. \tag{3.32}$$

From the transformation law (3.3), and the transformations

$$\begin{aligned}
\delta K_1 &= (\bar{\epsilon}\cdot\xi + \bar{\xi}\cdot\epsilon)K_1\,,\\
\delta K_2 &= \left[a\bar{z} + \bar{a}z - \bar{\epsilon}_1(\xi^1 - z\xi^2) - (\bar{\xi}_1 - \bar{z}\bar{\xi}_2)\epsilon^1\right]K_2\,,
\end{aligned} \tag{3.33}$$

one can show that

$$\delta\Phi^{(N,M)} = 2\left\{N[\bar{a}z - \bar{\epsilon}_1(\xi^1 - z\xi^2)] - M(\bar{\epsilon}\cdot\xi)\right\}\Phi^{(N,M)}\,. \tag{3.34}$$

The next step is to observe that, in the basis (3.26),

$$\begin{aligned}
\bar{\nabla}^1 &= -K_1\left\{\left[1 - \bar{\xi}_1(\xi^1 - z\xi^2)\right]\partial_{\bar{\xi}_1} - \left[z + \bar{\xi}_2(\xi^1 - z\xi^2)\right]\partial_{\bar{\xi}_2}\right\},\\
\bar{\nabla}^2 &= -K_2^{-1}\left(\partial_{\bar{\xi}_2} + \bar{z}\partial_{\bar{\xi}_1}\right)\,,
\end{aligned} \tag{3.35}$$

while $\nabla_{\bar{z}} \sim \partial_{\bar{z}_{sh}}$. Thus, in the new basis the chirality constraint (3.21a) becomes

$$\partial_{\bar{\xi}_1}\Phi^{(N,M)} = \partial_{\bar{\xi}_2}\Phi^{(N,M)} = 0 \quad \Rightarrow \quad \Phi^{(N,M)} = \Phi^{(N,M)}(\zeta_L)\,. \tag{3.36}$$

The chiral basis also simplifies the covariant analyticity condition $\mathcal{D}^+\Psi = 0$ that can be imposed on a chiral superfield $\Psi^{(N,M)}$ because it implies

$$\nabla_{\bar{z}}\Phi^{(N,M)}(\zeta_L) = 0 \quad \Rightarrow \quad \Phi^{(N,M)} = \Phi^{(N,M)}(z,\xi^i)\,. \tag{3.37}$$

One might describe this state of affairs by saying that the operator $\mathcal{D}^+$ is 'short' in the chiral basis, in which case it is worth noting, in contrast, that $\mathcal{D}_+$ does not share this property because

$$\begin{aligned}&\mathcal{D}_+\Psi^{(N,M)} = K_1^{M+1/2}K_2^{-(N-1)}\nabla_z^{(N)ch}\Phi^{(N,M)}\,,\\&\nabla_z^{(N)ch} = \partial_z - 2iN\mathcal{A}_z = \partial_z - 2N\partial_z \log K_2 = \partial_z - 2N\,\frac{\bar{z}_{sh}}{1+z\bar{z}_{sh}}\,.\end{aligned}\tag{3.38}$$

The possibility of imposing the further conditions (3.23) or (3.24) on chiral superfields reflects the existence of the two invariant subspaces

$$\text{(a)}\ \ (z,\ \xi^1 - z\xi^2) \quad \text{and} \quad \text{(b)}\ \ (\bar{z}_{sh},\ \xi^2 + \bar{z}_{sh}\xi^1)\,.\tag{3.39}$$

The $SU(2|1)$ invariance can be established by noting that

$$\begin{aligned}\delta(\xi^1 - z\xi^2) &= \epsilon^1 - z\epsilon^2 + \bar{a}z(\xi^1 - z\xi^2)\,,\\ \delta(\xi^2 + \bar{z}_{sh}\xi^1) &= \epsilon^2 + \bar{z}_{sh}\epsilon^1 + a\bar{z}_{sh}(\xi^2 + \bar{z}_{sh}\xi^1)\,,\end{aligned}\tag{3.40}$$

and using the transformations of z and $\bar{z}_{sh}$ given in (2.15) and (3.28). These subspaces can be identified with $CP^{(1|1)}$, which is a (holomorphic) supersphere [10], and its dual, the anti-holomorphic supersphere.

Finally, let us see how the more stringent set of conditions (3.23) with $M = 0$ is transformed into a constraint on the ξ^i dependence of $\Phi^{(N,M)}(z,\xi^i)$ defined in (3.37). At $M = 0$ the connection term drops out from $\mathcal{D}_2$, and we have

$$\begin{aligned}&\mathcal{D}_2\Psi^{(N,0)} = D_2\Psi^{(N,0)} = K_2^{-N-1/2}\nabla_2\Phi^{(N,0)}\,,\\&\nabla_2 = z\partial_{\xi^1} + \partial_{\xi^2}\,.\end{aligned}\tag{3.41}$$

Thus the extra condition in (3.23) is reduced to

$$\nabla_2\Phi^{(N,0)}(z,\xi^i) = 0 \quad \Rightarrow \quad \Phi^{(N,0)} = \Phi^{(N,0)}(z,\xi^1 - z\xi^2)\,.\tag{3.42}$$

4. Super-flag quantum mechanics

We now aim to formulate the dynamics of a particle on SF. We shall see that this leads naturally to superfields of the type described above. We begin by re-interpreting the 1-forms $(E^A, \mathcal{A}, \mathcal{B})$ as the corresponding 1-forms induced on the particle's worldline. Thus, we now have

$$E^A = dt\,\omega^A\,, \qquad \omega^A \equiv \dot{z}E_z^A + \dot{\xi}^i E_i{}^A\tag{4.1}$$

and

$$\mathcal{A} = dt\, A\,, \qquad A \equiv \left[\dot{z}\mathcal{A}_z + \dot{\xi}^i \mathcal{A}_i\right] + c.c.\,,$$
$$\mathcal{B} = dt\, B\,, \qquad B \equiv \dot{\xi}^i \mathcal{B}_i + c.c.\,. \tag{4.2}$$

Note the absence of a $\dot{z}$-term in B. The coefficients $\omega^A = (\omega^+, \omega^1, \omega^2)$ can be used to construct $SU(2|1)$-invariant kinetic terms, but a term quadratic in ω^i would be a 'higher-derivative' term that would effectively double the number of fermion variables. Fortunately, there is no need to include such a term; we may construct an $SU(2|1)$ invariant kinetic term from ω^+ alone. Although it also contains terms with derivatives of the 'fermi' variables ξ^i, these occur only in nilpotent 'fermion'-bilinear terms. Specifically,

$$\omega^+ = \dot{z}\omega + \dot{\xi}^i \omega_i\,, \tag{4.3}$$

where

$$\omega = K_1^{-\frac{1}{2}} K_2^{-1}\,,$$
$$\omega_1 = -K_1^{-\frac{3}{2}} K_2^{-1}\left(\bar{\xi}_2 + z\bar{\xi}_1\right),$$
$$\omega_2 = K_1^{-\frac{3}{2}} K_2^{-1} z \left(\bar{\xi}_2 + z\bar{\xi}_1\right). \tag{4.4}$$

Note that ω happens to be real, although all other coefficients are complex. We will see soon that the presence of the $\dot{\xi}^i$ terms in ω^+ is innocuous. In addition to the kinetic term, there are two possible WZ terms that we may construct from A and B. We record here that

$$\mathcal{A}_z = -iK_2^{-1}\left[\bar{z} - \xi^2\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\right],$$
$$\mathcal{A}_1 = -iK_2^{-1}\left(\bar{\xi}_1 - \bar{z}\xi_2\right),$$
$$\mathcal{A}_2 = iK_2^{-1} z\left(\bar{\xi}_1 - \bar{z}\xi_2\right), \tag{4.5}$$

and

$$\mathcal{B}_i = -iK_1^{-1}\bar{\xi}_i\,. \tag{4.6}$$

These considerations lead us to consider the Lagrangian

$$L = |\omega^+|^2 + NA + MB\,, \tag{4.7}$$

where N and M are two constants. Let (p, π_1, π_2) be the variables canonically conjugate to (z, ξ^1, ξ^2). An alternative, phase-space, Lagrangian is then

$$L = \left\{\left[\dot{z}p + i\dot{\xi}^i \pi_i + \lambda^i \varphi_i\right] + c.c.\right\} - H\,, \tag{4.8}$$

where H is the Hamiltonian

$$H = \omega^{-2}|p - N\mathcal{A}_z|^2, \tag{4.9}$$

and λ^i $(i = 1, 2)$ is a pair of complex Grassmann-odd Lagrange multipliers for the complex Grassmann-odd constraints $\varphi_i \approx 0$, where

$$\varphi_i = \pi_i + i\omega^{-1}\omega_i\,(p - N\mathcal{A}_z) + iN\mathcal{A}_i + iM\mathcal{B}_i\,. \tag{4.10}$$

Taken together with their complex conjugates, these constraints are second class, in Dirac's terminology. However, they are first class if viewed as two holomorphic constraints. Following the 'Gupta-Bleuler' method of dealing with complex second class constraints, as recently explained in the context of CSQM models in [10,11], we may view the constraints $\varphi_i \approx 0$ as gauge-fixing conditions for gauge invariances generated by their complex conjugates $\bar\varphi^i$. Stepping back to the gauge-unfixed theory, we may then quantize initially *without constraint* by setting

$$p = -i\,\frac{\partial}{\partial z}\,, \qquad \bar p = -i\,\frac{\partial}{\partial \bar z} \tag{4.11}$$

and

$$\pi_i = \frac{\partial}{\partial \xi^i}\,, \qquad \bar\pi^i = \frac{\partial}{\partial \bar\xi_i}. \tag{4.12}$$

The constraint functions $\bar\varphi^i$ then become the complex operators

$$\hat{\bar\varphi}^i = \frac{\partial}{\partial \bar\xi_i} - \omega^{-1}\bar\omega^i\left[\frac{\partial}{\partial \bar z} + N\,\frac{\partial \log K_2}{\partial \bar z}\right] + N\,\frac{\partial \log K_2}{\partial \bar\xi_i} - M\,\frac{\partial \log K_1}{\partial \bar\xi_i}\,. \tag{4.13}$$

To take the constraints into account it is now sufficient to impose the physical state conditions

$$\hat{\bar\varphi}^i|\Psi\rangle = 0 \qquad (i = 1, 2)\,. \tag{4.14}$$

We will solve this constraint in two steps. The first step, suggested by (3.30), is to set

$$\Psi = K_1^M K_2^{-N}\Phi \tag{4.15}$$

for 'reduced' wavefunction Φ, for which the physical state conditions are

$$\left(\frac{\partial}{\partial \bar\xi_i} - \omega^{-1}\bar\omega^i\frac{\partial}{\partial \bar z}\right)\Phi = 0 \qquad (i = 1, 2)\,. \tag{4.16}$$

These are equivalent to the two conditions

$$\left[\frac{\partial}{\partial\bar{\xi}_2} + \bar{z}\left(\frac{\partial}{\partial\bar{\xi}_1}\right)\right]\Phi = 0\,, \qquad \left[\frac{\partial}{\partial\bar{\xi}_1} - K_1^{-1}\left(\xi^2 + \bar{z}\xi^1\right)\frac{\partial}{\partial\bar{z}}\right]\Phi = 0\,. \quad (4.17)$$

These conditions are equivalent to the chirality conditions

$$\bar{D}^i\Phi = 0 \quad \text{or} \quad \bar{\nabla}^i\Phi = 0\,, \qquad (i = 1,2) \quad (4.18)$$

where $\bar{\nabla}^i$ were defined in (3.32). In other words, the reduced wavefunction is 'chiral', with N and M being two $U(1)$ charges. The general solution of such chirality constraints was given in (3.36),

$$\Phi = \tilde{\Phi}\left(z, \bar{z}_{sh}, \xi^1, \xi^2\right)\,, \quad (4.19)$$

where $\bar{z}_{sh}$ is the 'shifted' coordinate defined in (3.27). The function $\tilde{\Phi}$ can be expanded in a terminating Taylor series in ξ^1, ξ^2. Each of the four independent coefficient functions is determined by a single function on S^2, two of which are Grassmann-odd and two Grassmann-even.

The $SU(2|1)$ invariance of our model implies the existence of corresponding Noether charges. In particular, there exist Grassmann-odd Noether charges which, upon quantization become the operators

$$\begin{aligned}
\hat{S}_1 &= \frac{\partial}{\partial\xi^1} + M\bar{\xi}_1 - \bar{\xi}_1\left(\bar{\xi}\cdot\frac{\partial}{\partial\bar{\xi}}\right) + \left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\bar{z}\partial_{\bar{z}} + N\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right),\\
\hat{S}_2 &= \frac{\partial}{\partial\xi^2} + M\bar{\xi}_2 - \bar{\xi}_2\left(\bar{\xi}\cdot\frac{\partial}{\partial\bar{\xi}}\right) + \left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\partial_{\bar{z}}\,,\\
\hat{\bar{S}}^1 &= \frac{\partial}{\partial\bar{\xi}_1} + M\xi^1 + \xi^1\left(\xi\cdot\frac{\partial}{\partial\xi}\right) - \left(\xi^1 - z\xi^2\right)z\partial_z + N\left(\xi^1 - z\xi^2\right),\\
\hat{\bar{S}}^2 &= \frac{\partial}{\partial\bar{\xi}_2} + M\xi^2 + \xi^2\left(\xi\cdot\frac{\partial}{\partial\xi}\right) - \left(\xi^1 - z\xi^2\right)\partial_z\,,
\end{aligned} \quad (4.20)$$

These operators weakly anticommute with the constraints (4.10). Their non-zero anticommutation relations are

$$\begin{aligned}
&\{\hat{S}_i, \hat{S}_k\} = \{\hat{\bar{S}}^i, \hat{\bar{S}}^k\} = 0\,,\\
&\{\hat{S}_1, \hat{\bar{S}}^1\} = J_3 + B\,, \quad \{\hat{S}_2, \hat{\bar{S}}^2\} = B\,,\\
&\{\hat{S}_1, \hat{\bar{S}}^2\} = -J_+\,, \quad \{\hat{S}_2, \hat{\bar{S}}^1\} = J_-\,,
\end{aligned} \quad (4.21)$$

where

$$
\begin{aligned}
B &= z\partial_z - \bar{z}\partial_{\bar{z}} + \left(\xi^1\frac{\partial}{\partial\xi^1} - \bar{\xi}_1\frac{\partial}{\partial\bar{\xi}_1}\right) + 2M\,, \\
J_3 &= \left(\xi^2\frac{\partial}{\partial\xi^2} - \xi^1\frac{\partial}{\partial\xi^1}\right) - \left(\bar{\xi}_2\frac{\partial}{\partial\bar{\xi}_2} - \bar{\xi}_1\frac{\partial}{\partial\bar{\xi}_1}\right) - 2(z\partial_z - \bar{z}\partial_{\bar{z}}) + 2N\,, \\
J_+ &= \partial_z + (\bar{z})^2\partial_{\bar{z}} + \left(\xi_2\frac{\partial}{\partial\xi_1} - \bar{\xi}_1\frac{\partial}{\partial\bar{\xi}_2}\right) + \bar{z}N\,, \\
J_- &= \partial_{\bar{z}} + z^2\partial_z - \left(\xi_1\frac{\partial}{\partial\xi_2} - \bar{\xi}_2\frac{\partial}{\partial\bar{\xi}_1}\right) - zN\,.
\end{aligned}
\tag{4.22}
$$

Acting on the coordinates $Z = (z, \bar{z}, \xi^i, \bar{\xi}_i)$ these operators generate the transformations (2.15), and hence the transformations (3.3) of $\Psi(Z)$ for a superfield with $U(1)$ charges M and N.

5. Super-Landau levels

We have just seen that the wavefunction of a particle on SF is a chiral superfield. Of course, a general wavefunction will also be time-dependent but it can be expanded on a basis of stationary states with time-dependent coefficients that depend on the energy eigenvalues. These stationary states are time-independent chiral superfields, and our next task is to determine the energy eigenvalues and also the type of chiral superfield at each level. As we shall see, the ground state chiral superfield is one for which the reduced wavefunction is analytic.

Using the correspondence (4.11) we have

$$
i(p - N\mathcal{A}_z) \to \nabla_z^{(N)} = \partial_z - iNA_z\,. \tag{5.1}
$$

The quantum Hamiltonian operator $\hat{H}$ corresponding to the classical Hamiltonian (4.9) involves the product of $\nabla_z^{(N)}$ and its complex conjugate $\nabla_{\bar{z}}^{(N)}$. The product is ambiguous because, from (3.7), (3.18), (3.20),

$$
\left[\mathcal{D}_+, \mathcal{D}^+\right] = K_2^2 K_1 \left[\nabla_z^{(N)}, \nabla_{\bar{z}}^{(N)}\right] = 2N\,. \tag{5.2}
$$

The natural resolution of this ambiguity is to define the quantum Hamiltonian operator to be

$$
\hat{H} = -\frac{1}{2}\left\{\mathcal{D}_+, \mathcal{D}^+\right\} = -\frac{1}{2}\,K_2^2 K_1\{\nabla_z^{(N)}, \nabla_{\bar{z}}^{(N)}\}\,, \tag{5.3}
$$

as this is manifestly positive definite. Equivalently,

$$
\hat{H} = H_N = -\mathcal{D}_+\mathcal{D}^+ + N \equiv -K_2^2 K_1 \nabla_z^{(N)}\nabla_{\bar{z}}^{(N)} + N\,. \tag{5.4}
$$

One can show that

$$[\hat{H}, \hat{\bar{\varphi}}^i] = 0\,, \tag{5.5}$$

so that the Hamiltonian can be consistently restricted to a 'reduced Hamiltonian' operator

$$\hat{H}_{red} = K_1^{-M} K_2^N \, \hat{H} \, K_1^M K_2^{-N} = -\nabla_z^{(N+1)ch} \nabla_{\bar{z}}^{(N)ch} + N = -K_2^2 K_1 \nabla_z^{(N)ch} \partial_{\bar{z}} + N\,, \tag{5.6}$$

which acts on reduced wavefunctions $\Phi(z, \bar{z}_{sh}, \xi^1, \xi^2)$ (see (3.32) and (3.38) for the definition of $\nabla_{\bar{z}}^{(N)ch}$ and $\nabla_z^{(N)ch}$). Clearly, any holomorphic chiral superfunction $\Phi_0(z, \xi^1, \xi^2)$ is an eigenfunction of H_{red} with eigenvalue N. This is the ground state energy, although we postpone the proof of this until we complete, in the next section, the characterization of all admissible states, which also involves a determination of the degeneracies. First we must determine the energy levels, which we do using Schroedinger's factorization method. This method was recently applied to another supersymmetric extension of the Landau model for a particle on the 2-sphere [13] and what follows here is similar.

Using (3.18), we can rewrite (5.4) as

$$H_N = -UV + N\,, \tag{5.7}$$

where

$$U = K^{\frac{1}{2}} K_2 \nabla_z^{(N+1)}, \qquad V = K^{\frac{1}{2}} K_2 \nabla_{\bar{z}}^{(N)}\,. \tag{5.8}$$

The factorization trick exploits the fact that all non-zero eigenvalues of H_N are also eigenvalues of the 'reverse-order' Hamiltonian

$$\tilde{H}_N = -VU + N\,. \tag{5.9}$$

It follows that the first excited state of $\hat{H} = H_N$ is the ground state of $\tilde{H}_N$. However, we also have

$$\begin{aligned} \tilde{H}_N &= -K_1 K_2 \nabla_{\bar{z}}^{(N)} K_2 \nabla_z^{(N+1)} + N \\ &= -K_1 K_2^2 \nabla_{\bar{z}}^{(N+1)} \nabla_z^{(N+1)} + N \\ &= -K_1 K_2^2 \nabla_z^{(N+1)} \nabla_{\bar{z}}^{(N+1)} + 2(N+1) + N\,, \end{aligned} \tag{5.10}$$

where we have used (5.2) to get to the last line. Thus

$$\tilde{H}_N = H_{N+1} + 2N + 1\,. \tag{5.11}$$

We know that the ground state energy of H_{N+1} is $N+1$ so we deduce that the first excited state of $\hat{H} = H_N$ has energy $3N+2$. The corresponding eigenstate is $\Psi_1 = U\Psi_0 = \mathcal{D}_+\Psi_0$, where Ψ_0 is a ground state wavefunction.[c]

By iteration one now deduces that the full set of energy levels are

$$E = (2\ell+1)N + \ell(\ell+1)\,, \qquad \ell = 0,1,2,\ldots \tag{5.12}$$

with the corresponding reduced wavefunctions

$$\Phi^N_{(\ell)} = \nabla_z^{(N+\ell)ch} \ldots \nabla_z^{(N+1)ch}\Phi_0\left(z;\xi^1,\xi^2\right)\,, \qquad \ell = 1,2,\ldots\,, \tag{5.13}$$

where Φ_0 (having the $U(1)$ charges $(M-\ell/2, N+\ell)$) is a ground state reduced wavefunction.

It is worth noting that the easiest way to check that (5.12) is indeed the eigenvalue of the original hamiltonian H_N of (5.4), corresponding to the reduced wavefunction (5.13), is to consider the covariantly chiral wavefunction

$$\Psi^{(N,M)}_{(\ell)} = K_1^M K_2^{-N}\Phi^N_{(\ell)} = (\mathcal{D}_+)^\ell\Psi^{(N+\ell,M-\ell/2)}_{(0)}\,, \tag{5.14}$$

where here we make explicit the $U(1)\times U(1)$ charges of $\Psi_{(0)} = K_1^{M-\ell/2}K_2^{-(N+\ell)}\Phi_0$. Acting with $H_N = -\mathcal{D}_+\mathcal{D}^+ + N$ on this wavefunction, taking into account the $U(1)\times U(1)$ charges of $\mathcal{D}_+$, the commutation relation (3.7), and the covariant analyticity condition

$$\mathcal{D}^+\Psi^{(N+\ell,M-\ell/2)}_{(0)} = 0\,,$$

it is a matter of simple algebra to show that

$$H_N\Psi^{(N,M)}_{(\ell)} = [(2\ell+1)N + \ell(\ell+1)]\Psi^{(N,M)}_{(\ell)}\,. \tag{5.15}$$

Note the absence of any M-dependence of these eigenvalues. This makes it appear that the $U(1)$ charge M does not influence the structure of the Hilbert space. As we shall soon see, this is far from true.

6. Degeneracies

We now turn to a consideration of the $SU(2|1)$ content of the Hilbert space, which involves consideration of the Hilbert space norm. The $SU(2|1)$-invariant norm $||\Psi||$ of Ψ is given by the formula

$$||\Psi||^2 = \int d\mu\,|\Psi|^2 = \int d\mu_0\,K_2^{-2}|\Psi|^2\,, \tag{6.1}$$

[c] This covariantly chiral wavefunction has the $U(1)$ charges $(M-1/2, N+1)$ and so corresponds to the ground state of *another* system, with the coefficients $(M-1/2, N+1)$ in the relevant WZ terms.

where

$$\int d\mu_0 = \int d^2 z \prod_{i=1}^{2} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \bar{\xi}_i} \tag{6.2}$$

and the integral is over all complex z (which covers the sphere except for the point at infinity that does not contribute to the value of the integral). This result follows from the fact that

$$\left(sdet\, E_M{}^A\right)\left(sdet\, E^{\bar{M}}{}_{\bar{A}}\right) = K_2^{-2}. \tag{6.3}$$

The $SU(2|1)$ invariance of the measure $d\mu = d\mu_0\, K_2^{-2}$ can be verified using the transformation law (3.33) for K_2, and

$$\begin{aligned} \delta\left(d\mu_0\right) &= \left(\partial_z \delta z - \partial_{\xi^i} \delta \xi^i + \text{c.c.}\right) d\mu_0 \\ &= 2[\bar{a} z + a\bar{z} - \bar{\epsilon}_1(\xi^1 - z\xi^2) - (\bar{\xi}^1 - \bar{z}\bar{\xi}^2)\epsilon^1]\, d\mu_0\,. \end{aligned} \tag{6.4}$$

For a physical wavefunction of the form (4.15), we have

$$||\Psi||^2 = \int d\mu_0\, K_1^{2M} K_2^{-2(N+1)} |\Phi|^2. \tag{6.5}$$

As we saw in Section 3, for chiral Ψ the reduced wavefunction Φ takes the form

$$\Phi = \tilde{\Phi}\left(z, \bar{z}_{sh}, \xi^1, \xi^2\right) \tag{6.6}$$

where $\bar{z}_{sh}$ is the ‘shifted’ coordinate defined in (3.27).

We first evaluate (6.5) for the ground state wavefunction Ψ_0 for which Φ is analytic and has the component field expansion

$$\Phi\left(z, \xi^i\right) = A(z) + \xi^i \psi_i(z) + \xi^1 \xi^2 F(z)\,. \tag{6.7}$$

Using this in (6.5) and performing the Berezin integrals, we find that

$$\begin{aligned} ||\Psi_0||^2 = 2 \int \frac{dz d\bar{z}}{(1+z\bar{z})^{2(N+1)}} \Bigg[& M\left(2M+2N+1\right)|A|^2 + \frac{1}{2}|F|^2 \\ & + M\left(\bar{\psi}^1\psi_1 + \bar{\psi}^2\psi_2\right) + \frac{N+1}{1+z\bar{z}}\left(\bar{\psi}^2 + \bar{z}\bar{\psi}^1\right)\left(\psi_2 + z\psi_1\right) \Bigg]. \end{aligned} \tag{6.8}$$

For non-zero M we see that the ground-state multiplet contains two complex bosonic fields $A(z)$ and $F(z)$, as well as an SU(2) doublet of holomorphic Grassmann-odd fields $\psi_i(z)$ $(i = 1, 2)$. For these to be globally defined on the sphere, their norms should be square-integrable on S^2, i.e. the corresponding pieces of the integral on the right hand side of (6.8) should converge. This requires $A(z)$, $F(z)$ and each of the $\psi_i(z)$ to be polynomials of degree $\leq 2N$, which means that they each carry a $(2N{+}1)$-dimensional, spin N, representation of $SU(2)$. Actually, as $\psi_i(z)$ form an $SU(2)$ doublet, the Grassmann-odd fields carry the reducible representation $[\mathbf{2}] \otimes [\mathbf{2N+1}] = [\mathbf{2N+2}] \oplus [\mathbf{2N}]$ (the last term in (6.8) just involves the irreducible $[\mathbf{2N+2}]$ part of this $SU(2)$ representation). Thus we have a total of $4N+2$ bosonic components carried by $A(z)$ and $F(z)$ and $4N+2$ fermionic components carried by $\psi_i(z)$. Their transformation rules under the $U(1)$ charge B are specified by the external overall B-charge M and the transformation properties (2.17) of the coordinates (z, ξ^i).

From this result it is clear that $2N$ must be a positive integer, as expected because this was true of the bosonic Landau model. It then follows that $M \geq 0$ since the norm of the wavefunction with $\Phi = A(z)$ would otherwise be negative. For $M = 0$ this wave-function has zero norm. In this case the multiplet (6.7) splits into a semi-direct sum of an irreducible multiplet, with fields

$$F(z),\ \chi(z)\,, \qquad (\chi \equiv \psi_2 + z\psi_1)\,, \tag{6.9}$$

and a quotient which transforms into this irreducible set. In other words, for $M = 0$ we are facing a representation of $SU(2|1)$ that is not-fully reducible.[d] Normalizability implies that $F(z)$ is a (Grassmann-even) polynomial of degree $\leq 2N$, and that $\chi(z) = \psi_2(z) + z\psi_1(z)$ is a (Grassmann-odd) polynomial of degree $\leq 2N+1$. The $SU(2)$ content in this case is therefore $[\mathbf{2N+1}] \oplus [\mathbf{2N+2}]$ and these combine to yield the degenerate, 'superspin' $\left(N + \frac{1}{2}\right)$, irrep of $SU(2|1)$, of the type carried by a LLL particle on the supersphere [10].

Now we turn to the case of a general chiral superfield wavefunction, for

[d] This property is reflected in the structure of the transformation law (3.34) because the 'weight' piece at $M = 0$ becomes a function of the coordinates $(z, \xi^1 - z\xi^2)$, which form a closed set under the action of $SU(2|1)$; recall that precisely when $M = 0$ one can consistently impose on the holomorphic chiral superfield the additional Grassmann analyticity conditions (3.23), which forces it to 'live' on this smaller space. In terms of the component fields, this additional covariant constraint amounts to setting to zero the irreducible set $(F(z), \chi(z))$, after which the quotient becomes the degenerate irreducible $[\mathbf{2N+1}] \oplus [\mathbf{2N}]$, 'superspin' N, multiplet. Though the norm (6.8) is vanishing for the latter, one can presumably define for it an alternative $SU(2|1)$ invariant norm which is positive-definite (see [11]). We shall not dwell further on this possibility since it is unclear how to incorporate the conditions (3.23), (3.42) into our analyticity quantization method. Indeed, they inevitably require $\mathcal{D}^+ \approx 0$, which does not arise as a constraint within the hamiltonian formalism in our model, although it does in the Lowest Landau Level limit in which the kinetic term of $z, \bar{z}$ is suppressed in (4.7). So, this possibility would be of interest to study in the framework of Chern-Simons Quantum Mechanics on $SU(2|1)$.

which the reduced wavefunction depends both on z and on

$$\bar{z}_{sh} = \bar{z} - v, \qquad v \equiv \left(\xi^2 + \bar{z}\xi^1\right)\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right). \tag{6.10}$$

As H is nilpotent,

$$\Phi = \tilde{\Phi}\left(z, \bar{z}, \xi^i\right) - v\partial_{\bar{z}}\tilde{\Phi}\left(z, \bar{z}, \xi^i\right). \tag{6.11}$$

Using the component field expansion

$$\tilde{\Phi}\left(z, \bar{z}, \xi^i\right) = \tilde{A}(z, \bar{z}) + \xi^i\tilde{\psi}_i(z, \bar{z}) + \xi^1\xi^2\tilde{F}(z, \bar{z}) \tag{6.12}$$

we find that

$$||\Psi||^2 = ||\Psi||_0^2 - \int \frac{dzd\bar{z}}{(1+z\bar{z})^{2(N+1)}}\Bigg[(1+z\bar{z})^2|\partial_{\bar{z}}\tilde{A}|^2 + \left\{\left(\tilde{\bar{\psi}}^2 + \bar{z}\tilde{\bar{\psi}}^1\right)\left(\partial_{\bar{z}}\tilde{\psi}_1 - \bar{z}\partial_{\bar{z}}\tilde{\psi}_2\right) + h.c.\right\}\Bigg], \tag{6.13}$$

where $||\Psi||_0$ is the norm as it would be if we were dealing with the ground state (that is, the same as the ground state norm in (6.8) but with non-holomorphic component fields defined in (6.12)).

Notice the relative minus sign in (6.13). Let us see what effect this has on the first excited state wavefunction Ψ_1, for which

$$\tilde{\Phi} = \nabla_z{}^{(N+1)ch}\Phi\left(z, \xi^i\right). \tag{6.14}$$

In terms of the holomorphic component fields of the ground state reduced wavefunction the component fields of the first excited state are

$$\begin{aligned} A^{(1)} &= \left(\partial_z - \frac{2(N+1)\bar{z}}{1+z\bar{z}}\right)A(z), \\ \psi_i^{(1)} &= \left(\partial_z - \frac{2(N+1)\bar{z}}{1+z\bar{z}}\right)\psi_i(z), \\ F^{(1)} &= \left(\partial_z - \frac{2(N+1)\bar{z}}{1+z\bar{z}}\right)F(z). \end{aligned} \tag{6.15}$$

The derivatives with respect to $\bar{z}$ appearing in (6.13) are now trivially computed. After integrating by parts with respect to both ∂_z and $\partial_{\bar{z}}$ we arrive at the surprising result that the norm $||\Psi_1||^2$ coincides, up to a factor, with $||\Psi_1||_0^2$, which has the same form as (6.8) but with $M \to M - 1/2$, $N \to N + 1$. Thus the norm of the Ψ_1 is positive iff $M \geq 1/2$. Clearly, the norm of Ψ_0 is also positive under the same condition on M.

This result has the following generalization. The ℓth Landau level wavefunction has positive norm provided that

$$M \geq \ell/2\,. \tag{6.16}$$

It follows that for fixed M the physical Hilbert space is spanned by the states with

$$0 \leq \ell \leq 2[M]\,. \tag{6.17}$$

In other words, the number of Landau levels is *finite* in this model, a striking contrast with the bosonic problem for which the number of levels is infinite.

To prove this general result it is convenient to work with covariantly chiral wavefunctions, and we begin with the first level for which the corresponding covariantly chiral wavefunction is

$$\Psi^{(N,M)}_{(1)} = \mathcal{D}_+ \Psi^{(N+1,M-1/2)}_{(0)}\,, \qquad \mathcal{D}^+ \Psi^{(N+1,M-1/2)}_{(0)} = 0\,. \tag{6.18}$$

Substituting this into the norm as given in (6.1), and integrating by parts with respect to $\partial_{\bar{z}}$, it is easy to bring the norm into the form

$$||\Psi^{(N,M)}_{(1)}||^2 = -\!\int\! d\mu_0\, K_2^{-2}\ \overline{\Psi}_{(1)}{}^{(-N-1,-M+1/2)} \mathcal{D}^+ \mathcal{D}_+ \Psi^{(N+1,M-1/2)}_{(0)}. \tag{6.19}$$

Pulling $\mathcal{D}^+$ out to the right, using the commutation relation (3.7), taking into account the $U(1) \times U(1)$ charges and using the analyticity condition in (6.18), one deduces that

$$||\Psi^{(N,M)}_{(1)}||^2 = 2(N+1) \int d\mu_0\, K_2^{-2} \left|\Psi^{(N+1,M-1/2)}_{(0)}\right|^2 . \tag{6.20}$$

This differs from the ground-state norm by the factor $2(N+1)$ and the shift $(N, M) \to (N+1, M-1/2)$. This is just what we found before by direct evaluation in components.

The same method applied to the norm of the ℓ-level wavefunction (5.14), yields the result

$$||\Psi_{(\ell)}||^2 = \ell!\, \frac{(2N+\ell+1)!}{(2N+1)!} \int d\mu_0\, K_2^{-2} \left|\Psi^{(N+\ell,M-\ell/2)}_{(0)}\right|^2 . \tag{6.21}$$

Hence, up to the positive factor, this norm is given by the expression (6.8) with $M \to M - \ell/2$ and $N \to N + \ell$. From this, the bound (6.16) and the restriction (6.17) follow. In terms of the eigenvalue F of the $U(1)$ operator $\hat{F}$ commuting with $SU(2)$ and defined in (3.16), the restriction (6.16) is

$$\ell \leq \frac{1}{2}\, F - N\,. \tag{6.22}$$

Finally, we note that in the sector of all admissible states it is easy to show that N in (5.4) indeed provides the lowest energy. One sandwiches the first term in (5.4) between arbitrary physical states and finds that this average is always ≥ 0.

7. Concluding remarks

We have presented an $SU(2|1)$ invariant extension of the $SU(2)$-invariant Landau model for a particle on S^2, depending on $U(1)$ charges $2N$ and $2M$. In our case, the particle moves on the superflag manifold $SU(2|1)/[U(1) \times U(1)]$, which is a supermanifold of complex dimension $(1|2)$ having S^2 as its body. As was to be expected, the Hilbert superspace of each Landau level carries an irreducible representation of $SU(2|1)$, which depends on N, but, surprisingly, the number of admissible levels is finite, being determined by M.

Also notable is the fact that if $2M$ is an integer then the Hilbert superspace of the last admissible level (at $\ell = 2M$) carries a *degenerate* representation of $SU(2|1)$ corresponding to a wavefunction in a short supermultiplet. In particular, if $M = 0$ then only the lowest Landau level is admissible, and we effectively have a LLL model for a particle on the superflag, which defines a fuzzy superflag. One might have expected the $N \to \infty$ limit to yield a classical superflag but the $SU(2|1)$ content of its LLL Hilbert space coincides with the $SU(2|1)$ content of an LLL model for a particle on the supersphere, and this yields the classical supersphere in the large representation limit [10]).

Another notable feature, shared with the bosonic model, is that wavefunctions of any admissible Landau level for fixed N and M are expressed in terms of the ground state functions of a similar model, but with other values of these $U(1)$ charges. Since the ground states correspond to lowest Landau levels, and hence to some topological Chern-Simons mechanics, we deduce that the Hilbert space of the full Landau problem is the sum of Hilbert spaces for a set of inequivalent LLL models for a particle on $SU(2|1)/[U(1)\times U(1)]$.

As some avenues for further study, let us mention that we are not aware of any comparable analysis of the bosonic $SU(3)/[U(1) \times U(1)]$ 'Landau' model. One might also wish for a formulation that is manifestly independent of the parametrization of the coset (super)space, as can be achieved via the introduction of harmonic variables [15].

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