THE THREE-STRING VERTEX FOR A PLANE-WAVE BACKGROUND

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The three string vertex for Type IIB superstrings in a maximally supersymmetric plane-wave background can be constructed in a light-cone gauge string field theory formalism. The detailed formula contains certain Neumann coefficients, which are functions of a momentum fraction y and a mass parameter μ. This paper reviews the derivation of useful explicit expressions for these Neumann coefficients generalizing flat-space ($\mu = 0$) results obtained long ago. These expressions are then used to explore the large μ asymptotic behavior, which is required for comparison with dual perturbative gauge theory results. The asymptotic formulas, exact up to exponentially small corrections, turn out to be surprisingly simple.

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1. Introduction

A maximally supersymmetric plane-wave background in ten dimensions is an exact solution of Type IIB superstring theory [1]. Moreover, the string theory in this background is tractable, despite the fact that the background contains a nonvanishing RR field, provided that one uses the light-cone gauge Green–Schwarz (GS) formalism [2]. In that approach the world-sheet theory consists of free massive bosons and fermions. Thus it is trivial to read off the complete spectrum of the noninteracting theory.

The string states and their interactions are holographically dual to certain operators and their correlation functions in $\mathcal{N}=4$ super Yang-Mills theory in a suitable limit [3]. However, this paper will only consider the string side of the story and not discuss the duality.

The string interactions are encoded (using the formalism of light-cone-gauge string field theory) in a cubic interaction vertex. The three-string vertex has been formulated by Spradlin and Volovich [4, 5] and explored further by other authors [6]. These results generalize the flat-space light-cone gauge field theory results of [7, 8] to the plane-wave geometry. For recent reviews see [9–13].

This paper reviews work that makes the formulas for the Neumann coefficients that enter in the interaction vertex more explicit. These coefficients are defined in the first instance in terms of the inverse of a certain infinite dimensional matrix. The first step is to express this inverse matrix in terms of a certain infinite component vector [14, 15]. The next step is to derive an expression for this vector in terms of a certain scalar quantity and then to derive an explicit formula for the scalar [16]. Having obtained useful expressions for the Neumann coefficients (and hence the three superstring vertex), one can then explore the large μ (large curvature) limit, which is required for making contact with dual perturbative gauge theory computations.

2. Review of Basic Formulas

The type IIB superstring in the maximally supersymmetric plane-wave background is described in light-cone gauge by a free world-sheet theory. The eight bosonic and eight fermionic world-sheet fields each have mass μ, a parameter that enters in the description of the plane-wave geometry and the RR five-form field strength. The mass term has two important consequences. One is that it leads to a mixing of left-movers and right-movers. The other is that the zero modes are also described by harmonic oscillators of finite frequency. Altogether, a convenient labeling of the bosonic lowering and raising operators arising from quantization of the free world-sheet theory is a_m^I and $a_m^{I\dagger}$, where m runs from minus infinity to plus infinity and $I = 1, \ldots, 8$. These satisfy ordinary oscillator commutation relations

$$[a_m^I, a_n^{J\dagger}] = \delta_{mn}\delta^{IJ} . \tag{1}$$

There are also fermionic oscillators b_m^α and $b_m^{\alpha\dagger}$, which will not be discussed in this paper.

The spectrum of the free string theory is described by the light-cone Hamiltonian

$$H_2 = \sum_{m=-\infty}^{\infty} \omega_m N_m \, , \tag{2}$$

where N_m is the number of excitations of level m oscillators

$$N_m = \sum_{I=1}^{8} a_m^{I\dagger} a_m^I + \text{fermionic terms} \, , \tag{3}$$

and the frequencies are given by

$$\omega_m = \sqrt{m^2 + \mu^2\alpha^2} \, . \tag{4}$$

The second term in the square root is actually $(\alpha'\mu p_-)^2$, but we define $\alpha = \alpha' p_-$. (In flat space, we used the symbol p^+ rather than p_-, but in curved space a subscript is more natural, since momenta are conjugate to coordinates that are defined with superscripts.) The physical spectrum is given by the product of all the oscillator spaces subject to one constraint

$$\sum_{m=-\infty}^{\infty} m N_m = 0 \, . \tag{5}$$

In flat space this constraint reduces to the usual level-matching condition for left-movers and right-movers.

The three-string interaction vertex for type IIB superstrings in flat space was worked out in [7,8] and generalized to the plane-wave geometry in [4–6]. The formula can be written rather elegantly in terms of functionals, but to make its meaning precise and easily applicable to specific external states, it is desirable to expand it out in terms of oscillators. A convenient notation uses a tensor product of three string Fock spaces, labeled by an index $r = 1, 2, 3$. Then the three string interaction vertex contains a factor

$$|\, V_B \rangle = \exp\Big(\frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=-\infty}^{\infty} \sum_{I=1}^{8} a_{mr}^{I\dagger} \bar{N}_{mn}^{rs} a_{ns}^{I\dagger} \Big) |\, 0 \,\rangle \, . \tag{6}$$

The quantities $\bar{N}^{rs}_{mn}$, called Neumann coefficients, are the main objects of concern in this paper.[a] The three string vertex also contains a similar expression $|\,V_F\rangle$ made out of the fermionic oscillators and a "prefactor" that is polynomial in the various oscillators. We will not consider either of these in this paper. Suffice it to say that they are made out of the same basic objects, so that the results described here for the the bosonic factor are applicable to them as well.

In describing the Neumann matrices, it is convenient to consider separately the cases in which each of the indices m, n are either positive, negative or zero. Henceforth, the symbols m, n will always denote positive integers. Negative integers will be indicated by displaying an explicit minus sign. One result of [4], for example, using matrix notation for the blocks with positive indices, is

$$\bar{N}^{rs} = 1 - 2(C_r C^{-1})^{1/2} A^{(r)T} \Gamma_+^{-1} A^{(s)} (C_s C^{-1})^{1/2} \,. \tag{7}$$

Here $C_{mn} = m\delta_{mn}$ and $(C_r)_{mn} = \omega_{rm}\delta_{mn}$, where

$$\omega_{rm} = \sqrt{m^2 + (\mu\alpha_r)^2} \,. \tag{8}$$

The definitions of $A^{(r)}$ and Γ_+, and other expressions that appear here, are collected in Appendix A.

The blocks with both indices negative are related in a simple way to the ones with both indices positive by

$$\bar{N}^{rs}_{-m-n} = -\left(U_r \bar{N}^{rs} U_s\right)_{mn} \,, \tag{9}$$

where

$$U_r = C^{-1}(C_r - \mu\alpha_r) \,. \tag{10}$$

In the case of $\bar{N}^{33}$ these are the only nonvanishing terms. For the other Neumann coefficients the other nonvanishing terms are

$$\bar{N}^{rs}_{m0} = \bar{N}^{sr}_{0m} = \sqrt{2\mu\alpha_s}\,\epsilon^{st}\alpha_t \left[(C_r C^{-1})^{1/2} A^{(r)T} Y\right]_m, \quad r = 1,2,3\,, \quad s = 1,2\,, \tag{11}$$

$$\bar{N}^{rs}_{00} = (1 + \mu\alpha k)\,\epsilon^{rt}\epsilon^{su}\sqrt{\alpha_t\alpha_u}\,, \qquad r, s = 1,2\,, \tag{12}$$

$$\bar{N}^{r3}_{00} = \bar{N}^{3r}_{00} = -\sqrt{\alpha_r}\,, \qquad r = 1,2\,. \tag{13}$$

[a] Their definition here differs from that used in [7,8] by factors of $\sqrt{mn}$. The definition given here is more natural for the $\mu \neq 0$ generalization.

Here we have introduced (see Appendix A)

$$Y = \Gamma_+^{-1} B \,, \tag{14}$$

$$k = B^T \Gamma_+^{-1} B \,, \tag{15}$$

$$y = -\alpha_1/\alpha_3 \,, \tag{16}$$

and (setting $\alpha_3 = -1$)

$$\alpha = \alpha_1 \alpha_2 \alpha_3 = -y(1-y) \,. \tag{17}$$

The asymmetry between string number three and the other two strings is a reflection of the fact that the μ dependence of the formula breaks the cyclic symmetry that is present in the flat space case.

To make the formulas useful for comparison with the dual gauge theory, it would be helpful to have explicit formulas for the various matrix multiplications and inversions that appear. The quantities that we especially would like to evaluate explicitly are the matrix Γ_+^{-1}, the vector $Y = \Gamma_+^{-1} B$, and the scalar $k = B^T \Gamma_+^{-1} B$. In the case of flat space ($\mu = 0$) the results are known. Specifically

$$\bar{N}^{rs}_{mn} = -\frac{m\, n\, \alpha}{m\alpha_s + n\alpha_r}\, \bar{N}^r_m \bar{N}^s_n \qquad \text{for } \mu = 0 \,, \tag{18}$$

where

$$\bar{N}^r_m = \frac{\sqrt{m}}{\alpha_r}\, f_m(-\alpha_{r+1}/\alpha_r)\, e^{m\tau_0/\alpha_r} \qquad \text{for } \mu = 0 \,, \tag{19}$$

$$f_m(\gamma) = \frac{\Gamma(m\gamma)}{m!\,\Gamma(m\gamma + 1 - m)} \,, \tag{20}$$

and

$$\tau_0 = \sum_{r=1}^{3} \alpha_r \log|\alpha_r| \,. \tag{21}$$

In particular, still for $\mu = 0$, $\Gamma_+^{-1} = \frac{1}{2}(1 - \bar{N}^{33})$, $Y_m = -\bar{N}^3_m$, and $k = 2\tau_0/\alpha$.

3. Factorization Theorem

In this section we will derive the generalization of Eq. (18) that holds for the plane-wave geometry. The method of derivation is a fairly straightforward generalization of the one used for flat space in [7]. We begin by defining

$$\tilde{\Gamma}_+ = \sum_{r=1}^{3} A^{(r)} U_r^{-1} A^{(r)T} \,, \tag{22}$$

which differs from the Γ_+ by the replacement of U_r by U_r^{-1}. Then we consider the product

$$\Gamma_+ C^{-1}\,\tilde{\Gamma}_+ = \left(U_3 + \sum_1^2 A^{(r)} U_r A^{(r)T}\right) C^{-1} \left(U_3^{-1} + \sum_1^2 A^{(s)} U_s^{-1} A^{(s)T}\right). \tag{23}$$

Using various identities given in Appendix A, this simplifies to

$$\Gamma_+ C^{-1}\,\tilde{\Gamma}_+ = U_3 C^{-1}\tilde{\Gamma}_+ + \Gamma_+ C^{-1} U_3^{-1} - \frac{1}{2}\,\alpha_1\alpha_2 B B^T\,. \tag{24}$$

The next step is to use Eqs. (A.12) and (A.8) to deduce that

$$\tilde{\Gamma}_+ = \Gamma_+ + \mu\alpha B B^T\,. \tag{25}$$

Substituting this into the previous equation and multiplying left and right by Γ_+^{-1} gives

$$C^{-1} U_3^{-1} \Gamma_+^{-1} + \Gamma_+^{-1} U_3 C^{-1} = C^{-1} + \frac{1}{2}\,\alpha_1\alpha_2 Y Y^T + \mu\alpha Z Y^T\,, \tag{26}$$

where we have defined

$$Z = (1 - \Gamma_+^{-1} U_3) C^{-1} B\,. \tag{27}$$

The next step is to eliminate Z from Eq. (26). This is achieved by multiplying the equation on the right with the vector B. This gives a linear equation for Z, whose solution is

$$Z = \frac{1}{1+\mu\alpha k}\left(C^{-1}U_3^{-1} - \frac{1}{2}\alpha_1\alpha_2 k\right) Y\,. \tag{28}$$

Substituting this back into Eq. (26) and simplifying gives the formula

$$\{\Gamma_+^{-1}, C_3\} = C + \frac{1}{2}\,\frac{\alpha_1\alpha_2}{1+\mu\alpha k}\, C U_3^{-1} Y Y^T C U_3^{-1}\,. \tag{29}$$

In terms of components

$$(\Gamma_+^{-1})_{mn} = \frac{m}{2\omega_m}\,\delta_{mn} + \frac{y\,(1-y)(\omega_m-\mu)(\omega_n-\mu) Y_m Y_n}{2[1-\mu y(1-y)k](\omega_m+\omega_n)}\,, \tag{30}$$

where

$$\omega_m = \omega_{3m} = \sqrt{m^2+\mu^2}\,. \tag{31}$$

This result be recast as a formula for the Neumann coefficient matrix $\bar{N}^{33}_{mn}$. The result is

$$\bar{N}^{33}_{mn} = -\frac{mn\alpha_1\alpha_2}{1+\mu\alpha k}\,\frac{\bar{N}^3_m\bar{N}^3_n}{\omega_{3m}+\omega_{3n}}\,, \tag{32}$$

where

$$\bar{N}^3_m = -\left[(C^{-1}C_3)^{1/2}U_3^{-1}Y\right]_m\,. \tag{33}$$

Some further simple manipulations give the generalization [14, 15]

$$\bar{N}^{rs}_{mn} = -\frac{mn\alpha}{1+\mu\alpha k}\,\frac{\bar{N}^r_m\bar{N}^s_n}{\alpha_s\omega_{rm}+\alpha_r\omega_{sn}}\,, \tag{34}$$

where

$$\bar{N}^r_m = -\left[(C^{-1}C_r)^{1/2}U_r^{-1}A^{(r)T}Y\right]_m\,. \tag{35}$$

This is the desired generalization of the flat-space formula Eq. (18). However, we still require a generalization of Eq. (19) as well as an explicit formula for k. Note that combining Eq. (35) with Eq. (11) gives

$$\bar{N}^{rs}_{m0} = \bar{N}^{sr}_{0m} = -\sqrt{2\mu\alpha_s}\epsilon^{st}\alpha_t U_r\bar{N}^r_m\,, \qquad r=1,2,3\,, \quad s=1,2\,. \tag{36}$$

4. Determination of Y and k

To complete the explicit determination of the Neumann matrices, and thus the three string vertex, we need useful formulas for

$$(\Gamma_+^{-1})_{mn}\,, \qquad Y_m = (\Gamma_+^{-1}B)_m\,, \qquad k = B^T\Gamma_+^{-1}B$$

as functions of y and μ. In view of of Eq. (30), if we knew Y_m and k we would know $(\Gamma_+^{-1})_{mn}$. The strategy that we will use for obtaining them is to derive first-order differential equations (in μ) and input the known values at $\mu = 0$ as initial conditions. Using the various definitions and identities in Appendix A, Ref. [16] derived the differential equation

$$\frac{\partial Y_m}{\partial\mu} = \left[\frac{1}{2}\frac{\partial F}{\partial\mu}\left(1-\frac{\mu}{\omega_m}\right) - \frac{\mu}{\omega_m^2}\right]Y_m\,, \tag{37}$$

where

$$F(\mu,y) = \log\left[1-\mu y(1-y)k(\mu,y)\right]. \tag{38}$$

The derivation of Eq. (37) is sketched in Appendix B.

Equation(37) has the solution

$$Y_m(\mu, y) = \frac{m}{\omega_m} \exp\left[\frac{1}{2}\int_0^\mu \frac{\partial F}{\partial \mu}\left(1 - \frac{\mu}{\omega_m}\right) d\mu\right] Y_m(0, y)\,. \tag{39}$$

Thus, since $Y_m(0, y)$ is known, if we knew $F(\mu, y)$, we would know $k(\mu, y)$ and $Y_m(\mu, y)$ and hence all the Neumann coefficients. Since, we have one fewer equations than unknowns, we need to input one additional piece of information. One that is easy to obtain and does the job is the asymptotic formula

$$Y_m \sim \frac{m}{2\mu}\, B_m + O(\mu^{-2}) \tag{40}$$

for large μ. Combining this condition with Eq. (39) implies that

$$\exp\left\{\frac{1}{2}\int_0^\infty \frac{\partial F}{\partial \mu}\left(1 - \frac{\mu}{\omega_m}\right) d\mu\right\} = \frac{B_m}{2Y_m(0)}\,. \tag{41}$$

Taking the logarithm, integrating by parts, and substituting the known value of the right-hand side gives

$$\int_0^\infty (m^2 + \mu^2)^{-3/2} F(\mu, y)\, d\mu = G(m, y), \tag{42}$$

where

$$G(z, y) = \frac{2\tau_0}{z} + \frac{2}{z^2}\log\left(\frac{\Gamma(1+z)}{\Gamma(1+zy)\Gamma(1+z(1-y))}\right). \tag{43}$$

This formula must hold for $m = 1, 2, ...$ and $0 \le y \le 1$. The inverse integral transform, which does not seem to exist in the mathematical literature, determines $F(\mu, y)$. Ref. [16] proves that for a function $G(m, y)$ that is holomorphic in the right half m plane and vanishes at infinity in that half plane, the inverse integral transform that solves the integral equation Eq. (42) is

$$F(\mu, y) = -\frac{i\mu^2}{\pi}\int_0^\pi \cos\theta\; G(-i\mu\cos\theta, y)\, d\theta\,. \tag{44}$$

The proof is sketched in Appendix C. Using this, one can show that for our specific choice of $G(m, y)$ in Eq. (43)

$$F(\mu, y) = 2\mu\tau_0 + 2\sum_{r=1}^{3} \phi(\mu\alpha_r)\,, \tag{45}$$

where

$$\phi(x) = \sum_{n=1}^{\infty} \left[\log \left(\frac{\sqrt{n^2 + x^2} + x}{n} \right) - \frac{x}{n} \right]. \tag{46}$$

We now have sufficiently explicit formulas to construct large μ asymptotic expansions.[b] Large μ corresponds to small $\lambda' = 1/\mu^2$, which is the effective coupling constant in the dual gauge theory [3]. Differentiating Eq. (46) twice, introducing a contour integral (Sommerfeld-Watson) representation of the series, and integrating by parts one can show that

$$\phi''(x) = -x \sum_{n=1}^{\infty} \frac{1}{(x^2 + n^2)^{3/2}} = -\frac{1}{x} + \frac{1}{2x^2} - \pi \int_1^{\infty} \frac{z\, dz}{\sqrt{z^2 - 1}} \frac{1}{\sinh^2(\pi x z)}. \tag{47}$$

Integrating back, this allows us to deduce that

$$F(\mu, y) = -\ln[4\pi\mu y(1 - y)] + J(\mu y) + J(\mu(1 - y)) - J(\mu), \tag{48}$$

where

$$J(x) = \frac{2}{\pi} \int_1^{\infty} \frac{\ln(1 - e^{-2\pi x z})}{z\sqrt{z^2 - 1}}\, dz. \tag{49}$$

Note that $J \sim \exp(-2\pi x)$ for large x.

5. Asymptotic Behavior

We can now derive asymptotic results that include all inverse powers of μ and have leading corrections of order $\exp[-2\pi y\mu]$ (if $y \leq 1/2$). For example,

$$F(\mu, y) \approx -\ln[4\pi\mu y(1 - y)]. \tag{50}$$

The amazing thing is that a remarkably simply expression captures the result to all finite orders in $1/\mu$ and therefore one can easily read off predictions for the dual gauge theory that are valid (at this order of nonplanarity) to all orders in perturbation theory! The exponentially suppressed corrections encoded by the J functions correspond to nonperturbative effects in the dual gauge theory. What these effects are, however, is mysterious. They don't seem to be related to instantons or any other familiar nonperturbative phenomena.

[b] This analysis was initiated in [17] and completed in [16].

Combining Eqs. (50) and (38) one finds that

$$k(\mu, y) \approx \frac{1}{\mu y(1-y)} - \frac{1}{4\pi\mu^2 y^2 (1-y)^2}\,. \tag{51}$$

Similarly, substituting Eq. (50) in Eq. (39) gives

$$Y_m(\mu, y) \approx \sqrt{\frac{\mu + \omega_m}{2\mu}}\frac{m}{2\omega_m} B_m\,, \tag{52}$$

or in matrix notation

$$Y \approx \frac{1}{2\sqrt{2\mu}} U_3^{1/2} C^{3/2} C_3^{-1} B\,. \tag{53}$$

From this it follows that

$$\bar{N}_n^r \approx \frac{(-1)^{r(n+1)}}{2\pi y(1-y)} \sqrt{\frac{|\alpha_r|}{2\mu n \omega_{rn} U_{rn}}} s_{rn}\,, \qquad r \in \{1, 2, 3\}\,, \tag{54}$$

where

$$s_{1m} = s_{2m} = 1, \qquad s_{3m} = -2\sin(\pi m y). \tag{55}$$

The asymptotic expansions of the Neumann matrices are then given by substitution in Eqs. (34) and (36).

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Appendix A. Useful Definitions and Identities

The light-cone momenta in the minus direction that appear in the three-string vertex are defined to be α_r/α', $r = 1, 2, 3$. For the process in which string #3 splits into strings #1 and #2, we take $\alpha_1, \alpha_2 > 0$, $\alpha_3 < 0$. Momentum conservation implies that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. We also define the momentum fraction carried by string #1 to be $y = -\alpha_1/\alpha_3$, which satisfies $0 < y < 1$. It follows that $1 - y = -\alpha_2/\alpha_3$ is the momentum fraction carried by string #2. It is sometimes convenient to set $\alpha_3 = -1$, which can always be achieved by a suitable Lorentz boost. Only then does the mass parameter μ have an invariant meaning.

The matrices $A^{(r)}_{mn}$, which appear in the Neumann coefficients, are given by

$$A^{(1)}_{mn} = \frac{2}{\pi}\,(-1)^{m+n+1}\sqrt{mn}\,\frac{y\sin{(m\pi y)}}{n^2 - m^2y^2}\,, \tag{A.1}$$

$$A^{(2)}_{mn} = \frac{2}{\pi}\,(-1)^{m}\sqrt{mn}\,\frac{(1-y)\sin{(m\pi y)}}{n^2 - m^2(1-y)^2}\,, \tag{A.2}$$

and $A^{(3)}_{mn} = \delta_{mn}$. The indices m, n range from 1 to infinity. Additional quantities that we need are

$$B_m = \frac{2\alpha_3}{\pi\alpha_1\alpha_2}\,(-1)^m\,\frac{\sin{(m\pi y)}}{m^{3/2}} \tag{A.3}$$

and

$$C_{mn} = m\,\delta_{mn}\,. \tag{A.4}$$

These quantities are all μ independent and were defined already in the flat space analysis.

The infinite matrices $A^{(r)}_{mn}$ and the infinite vector B_m satisfy a number of useful relations, which we record here

$$A^{(r)T}CA^{(s)} = -\frac{\alpha_3}{\alpha_r}\,C\,\delta^{rs}\,, \qquad r, s = 1, 2\,, \tag{A.5}$$

$$A^{(r)T}C^{-1}A^{(s)} = -\frac{\alpha_r}{\alpha_3}\,C^{-1}\delta^{rs}\,, \qquad r, s = 1, 2\,. \tag{A.6}$$

The symbol T means matrix transpose. Some additional useful identities are

$$\sum_{r=1}^{3}\frac{1}{\alpha_r}A^{(r)}CA^{(r)T} = 0\,, \tag{A.7}$$

$$\sum_{r=1}^{3}\alpha_r A^{(r)}C^{-1}A^{(r)T} = \frac{\alpha}{2}BB^T\,, \tag{A.8}$$

where we have introduced

$$\alpha = \alpha_1\alpha_2\alpha_3\,. \tag{A.9}$$

Additional matrices that involve the mass parameter μ of the plane-wave geometry, introduced in [4], are

$$(C_r)_{mn} = \omega_{rm}\delta_{mn} = \sqrt{m^2 + \mu^2\alpha_r^2}\,\delta_{mn} \tag{A.10}$$

and

$$U_r = C^{-1}(C_r - \mu\alpha_r)\,. \tag{A.11}$$

Note that

$$(U_r)^{-1} = C^{-1}(C_r + \mu\alpha_r) = U_r + 2\mu\alpha_r C^{-1}\,. \tag{A.12}$$

A crucial construct is the infinite matrix

$$\Gamma_+ = \sum_{r=1}^{3} A^{(r)} U_r A^{(r)T}\,. \tag{A.13}$$

Explicit formulas for the inverse of Γ_+ are a main goal of our work. Related quantities that also are needed are the infinite vector

$$Y = \Gamma_+^{-1} B \tag{A.14}$$

and the scalar

$$k = B^T \Gamma_+^{-1} B\,. \tag{A.15}$$

Appendix B. Derivation of the Differential Equation

This appendix sketches the derivation of Eq. (37), which we copy here

$$\frac{\partial Y_m}{\partial \mu} = \left[\frac{1}{2}\,\frac{\partial F}{\partial \mu}\left(1 - \frac{\mu}{\omega_m}\right) - \frac{\mu}{\omega_m^2}\right] Y_m\,. \tag{B.1}$$

The matrix $\Gamma_+ = \sum_r A^{(r)} U_r A^{(r)\mathrm{T}}$, introduced in Appendix A, only depends on μ through the dependence of U_r on μ. Its derivative can be written in the form

$$\frac{\partial \Gamma_+}{\partial \mu} = -\frac{1}{2}\,\alpha B B^{\mathrm{T}} + \mu N\,, \tag{B.2}$$

where

$$N = \sum_{r=1}^{3} \alpha_r^2 A^{(r)} C^{-1} C_r^{-1} A^{(r)\mathrm{T}}\,. \tag{B.3}$$

It follows that

$$\frac{\partial Y}{\partial \mu} = \frac{1}{2}\,k\alpha Y - \mu \Gamma_+^{-1} N Y\,. \tag{B.4}$$

The product NY can be recast in the form

$$NY = g_1 C_3^{-2} B + g_2 B\,, \tag{B.5}$$

where we define the coefficients g_1 and g_2 to be the scalar quantities

$$g_1 = \frac{2\,(1+\mu\alpha k)}{2+\mu\alpha k+\mu^2\alpha k_1}\,, \tag{B.6}$$

$$g_2 = \left(\frac{\alpha}{2}\right)\frac{\alpha k^2+\mu\alpha k k_1+2k_1}{2+\mu\alpha k+\mu^2\alpha k_1}\,, \tag{B.7}$$

and

$$k_i = B^{\mathrm{T}} C_3^{-i} Y\,. \tag{B.8}$$

The above equations imply that

$$\frac{\partial k}{\partial \mu} = B^{\mathrm{T}}\frac{\partial Y}{\partial \mu} = \frac{1}{2}\,\alpha k^2 - \mu g_2 k - \mu g_1 k_2. \tag{B.9}$$

This is not very useful as it stands, since there is no other apparent way to determine k_2. (k_1 could be determined, but that will turn out not to be necessary.) Substituting the equation for NY and an identity for $[C_3^{-2}, \Gamma_+^{-1}]$ deduced from Eq. (29), one can recast Eq. (B.4) in the form

$$\frac{\partial Y}{\partial \mu} = (F_0 + F_1 C_3^{-1} + F_2 C_3^{-2})\,Y\,, \tag{B.10}$$

where the scalar functions F_i are given by

$$F_0 = \frac{1}{2}\,\alpha k - \mu g_2 + \frac{1}{2}\,\mu g_1\,\frac{\alpha}{1+\mu\alpha k}\,(k_1 - \mu k_2)\,, \tag{B.11}$$

$$F_1 = -\frac{1}{2}\,\mu g_1\,\frac{\alpha}{1+\mu\alpha k}\,(k - \mu^2 k_2)\,, \tag{B.12}$$

$$F_2 = -\mu g_1 + \frac{1}{2}\,\mu^2 g_1\,\frac{\alpha}{1+\mu\alpha k}\,(k - \mu k_1)\,. \tag{B.13}$$

Using the equations above to eliminate g_1, g_2, k_1, and k_2, we find

$$F_2 = -\mu, \qquad F_1 = -\mu F_0, \qquad F_0 = \frac{\alpha}{2}\,\frac{1}{1+\mu\alpha k}\,(k + \mu k')\,. \tag{B.14}$$

This allows us to rewrite Eq. (B.10) in the desired form Eq. (B.1).

Appendix C. The Integral Transform

The analysis in Sec. 4 of the text required solving the following integral equation for $f(x)$

$$g(w) = \int_0^\infty \frac{f(x)}{(x^2+w^2)^{3/2}}\,dx\,, \tag{C.1}$$

where $g(w)$ is a given function that is holomorphic in the right-half plane and vanishes at infinity in that half plane. We claim [16] that the unique solution is

$$f(x) = i\,\frac{x^2}{\pi}\int_0^\pi g(ix\cos\theta)\cos\theta\,d\theta\,. \tag{C.2}$$

The proof that elimination of f from Eqs. (C.1) and (C.2) gives $g = g$ is a consequence of elementary integration. This proves the existence of a solution. The proof of uniqueness requires that elimination of g should give $f = f$. After making changes of variables and deforming integration contours, one can argue that this requires the identity

$$\delta(y - y') = \frac{\sqrt{y}}{4\pi i}\int_C \frac{dz}{\sqrt{1+z}}\,\frac{1}{(zy + y')^{3/2}}\,, \tag{C.3}$$

where $y, y' > 0$. The contour can be taken to be the unit circle $|z| = 1$, in the counterclockwise sense, starting and ending at the point $z = -1$. It is an elementary application of Cauchy's theorem to show that this integral vanishes for $y < y'$ and for $y' < y$. That it is has the right singularity at $y = y'$ can be verified by showing that the Laplace transform of both sides give an identity.

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