R–SYMMETRIES, ALGEBRAIC GEOMETRY, AND AdS/CFT

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We study the flow of central charges in $\mathcal{N}=1$ supersymmetric gauge theories which admit large N supergravity duals. For theories lying in the conformal window, supersymmetry relates this evolution to a renormalization of the R-charges, and this information can, in many cases, be extracted from the dual geometry making use of the conserved $U(1)_S$ current along the flow. In particular, we argue that these charges are determined purely by the complex structure of the 6-dimensional manifold transverse to the D-branes which source the geometry. We demonstrate this relationship in a class of flows between A_k orbifolds and generalized conifolds.

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This paper is dedicated to the memory of Ian Kogan. Although I knew Ian rather briefly in comparison to many of his long-term friends and collaborators who are contributing to this volume, his enthusiasm for all things, not least physics, was infectious and struck a chord with me, as with so many others. It was always a highly stimulating and enjoyable experience discussing physics with Ian, and he will be sorely missed.

1. Introduction

Any comprehensive framework for classifying the phase structure and universality classes of quantum field theories in four dimensions will necessarily require a detailed understanding of the possible conformal fixed points. While little is known in general about such interacting conformal theories, supersymmetry provides important constraints and analytical tools which have led to greater insight into superconformal field theories (SCFTs). In particular, the existence of exact duality symmetries mapping between different regions in the space of marginal parameters has allowed insight into strong coupling dynamics, both in a given SCFT itself and in the nonconformal phases to which it flows under relevant perturbations.

The most basic characterization of a 4D CFT, as in two dimensions, involves a set of central charges which specify the singularity structure of two and three-point correlators of the energy-momentum tensor $T_{\mu\nu}$. These are conveniently encoded in the trace anomaly of $T_{\mu\nu}$ in a background geometry with metric $g_{\mu\nu}$,

$$16\pi^2 \langle T^\mu_\mu \rangle = c(C_{\mu\nu\rho\sigma})^2 - a(\tilde{R}_{\mu\nu\rho\sigma})^2 + a'\Box R \ , \qquad (1)$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, $\tilde{R}_{\mu\nu\rho\sigma}$ is the dual of the curvature, and a and c are the two renormalization scheme-independent central charges (a' is scheme-dependent and will not concern us here). It is believed that the charge a satisfies an irreversibility criterion [1] analogous to the Zamolodchikov c-theorem in 2D, although no general proof is known.

Certain conformal theories have a dual description in terms of IIB string theory on spaces whose noncompact part is AdS_5 [2–5], and this dual description has provided a fruitful calculational arena, particularly for those theories which possess a large-N limit and so the dual reduces to IIB supergravity on the AdS geometry. However, a dual supergravity calculation of the Weyl anomaly by Henningson and Skenderis [6] has shown that this dual description exists only if the field theory satisfies the constraint $a = c$ at large N. This result follows from the identification of the anomaly with the coefficient of the log-divergent term in the regularized volume of a generic asymptotically-AdS space. In 10D units, with $V(H_5)$ the volume of the transverse horizon,

$$a = c = \frac{L^5}{V(H_5)} , \tag{2}$$

where L is a generic scale associated with the 10D Planck mass that we will not need explicitly. The constraint $a = c$ is generic for any CFT whose dual reduces (in some limit) to supergravity on AdS_5, as is clear from the absence of more than one dimensionless combination of parameters. Note that this also applies to more general constructions where D3-branes are replaced by higher dimensional branes wrapped on cycles [7], or intersecting solutions [8], but of course need not hold beyond leading order in N [9].

From a field-theoretic point of view, imposing the condition $a = c$ at large N, which is equivalent to the absence of gravitational U(1) anomalies for the R-current, is a rather nontrivial constraint even for theories with $\mathcal{N}=2$ supersymmetry, and the structure of this class is still poorly understood [10]. It is therefore of interest to understand what characterizes this class of theories within the dual string theory, where we consider for example N D3-branes transverse to a 6-dimensional space C_6. In particular, to admit a dual SCFT, the space C_6 must be Calabi–Yau. Moreover, since for large N we consider the near-horizon region, interesting theories with less than maximal supersymmetry will arise at singular points of C_6. Provided the superconformal symmetry acts via isometries on C_6 it must also exhibit a $\mathbb{C}^*$-action, a "complexified renormalization group" action, combining dilatations with a $\mathrm{U}(1)_R$ symmetry. The Calabi–Yau metric on C_6 must therefore be conical, with a horizon H_5 possessing an Einstein–Sasaki metric, which in turn can be realized as a $\mathrm{U}(1)_R$ fibration over a Kähler–Einstein space B_4.

When the U(1) action is regular (i.e. acts freely), this structure is surprisingly rigid as there is a complete classification of Kähler–Einstein 4-manifolds with positive curvature. The relevant cases are: (1) $B_4 = \mathbb{CP}^2$, leading to $H_5 = S^5$ or $H_5 = S^5/\mathbb{Z}_3$; (2) $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$, with $H_5 = T^{11}$ or $H_5 = T^{11}/\mathbb{Z}_2$; and finally (3) B_4 is a more general del Pezzo surface, the blowup of $\mathbb{CP}^2$ at k points, with $3 \leq k \leq 8$. In the first two cases, the Einstein–Sasaki metric on H_5 is unique up to rescaling. This list is, however, rather incomplete as there is no overriding reason for ignoring spaces for which the U(1) action is not regular – i.e the U(1) orbits are not all of constant length. Thus we must also allow B_4 to be, for example, a Kähler–Einstein orbifold, and no general classification of the resulting non-regular (or more specifically quasi-regular, where the U(1) orbits are compact) Einstein–Sasaki manifolds is known. On one hand this conclusion is not helpful from the point of view of classification, but the recent discovery of several examples for $S^2 \times S^3$ [11,12] suggests that the the class may nevertheless be large enough to be of some interest.

From the relation (2), it is clear that knowledge of the horizon volume is sufficient to determine the central charges, a result that is independent of supersymmetry. However, for SCFTs we should expect additional constraints, and one may hope that the charges are calculable even if the specific metric, and thus the horizon volume, is not known. This is the idea that we will pursue here, relying on certain known results about the dual SCFTs and their perturbations. Specifically, at the fixed points supersymmetry relates the central (or dilatation) charges to R-charges, and the latter are often more easily calculated. Moreover, under perturbations, the R-current is anomalous, but there remains a particular combination, known as $\mathrm{U}(1)_S$, which (at least perturbatively) is conserved [13]. The charges under this current then determine the R-charges at any putative infra-red (IR) fixed point. We will argue that these generalized R-charges are often calculable in the dual string description purely from knowledge of the complex structure [a] of C_6 – a consequence of supersymmetry.

We will illustrate this by considering a particular class of supersymmetric dual flows in the conformal window – mass perturbations of $\mathcal{N}=2$ A_k orbifolds $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_{k+1}$, which flow to generalized A_k conifolds at the IR fixed point. The UV background is dual to an $\mathcal{N}=2$ "quiver" gauge theory with gauge group $\mathrm{U}(N)^{k+1}$ and bi-fundamental matter under each $\mathrm{U}(N)$ factor. Adding mass terms for each adjoint, subject to the constraint $\sum_i m_i = 0$, leads to an $\mathcal{N}=1$ IR SCFT dual to the generalized conifold. These flows cover two generic classes of Calabi–Yau singularities – namely quotient and hyperquotient singularities which admit a toric description, and thus exhibit the R-symmetries via their complex structure determined as hypersurfaces in $\mathbb{C}^4$. The dual flow of the A_1 orbifold to the conifold was first studied by Klebanov and Witten [17], and generalizations were considered by Gubser et al. [18] (see also [19–21]). Central charges were determined from the horizon volume [22,23] following (2), and we will reproduce these results purely from the complex structure of C_6. A 5D supergravity description for this class of fixed points has been discussed in [24,25].

The paper is organized as follows. In the following section, for completeness, we review the constraints imposed by supersymmetry on central

[a] An alternative approach, studying dibaryon operators, corresponding to D3-branes wrapped on 3-cycles, has also been utilized recently [14–16] to extract these R-charges from geometric data.

charges in SCFTs. In particular, we recall the existence, alluded to above, of a conserved $\mathrm{U}(1)_S$ R-current, and how the charges under this current may be used to determine the central charges at IR fixed points following Anselmi et al. [26,27]. We also discuss the recent analysis of Intriligator and Wecht [15] which identifies, via supersymmetry, the unique superconformal $\mathrm{U}(1)_R$ current via constraints on 't Hooft anomalies for all the non-anomalous currents. In Section 3, we turn to the dual string background and describe how the R-symmetry may be identified in terms of a scaling symmetry within the dual geometry. We also discuss the additional constraints on 't Hooft anomalies that arise via the embedding within string theory, and how these provide a restricted version of the constraints of Intriligator and Wecht which determine the superconformal R-current. In Section 4, we consider the specific class of dual backgrounds described above, determine the UV and IR R-charges directly from the complex structure of the geometry and obtain the evolution of the central charges. The results are consistent with existing results for the horizon volumes [23]. Section 5 contains some concluding remarks.

2. Supersymmetry, U(1) currents, and central charges

We will begin by reviewing the field-theoretic constraints on the central charges of SCFTs, restricting our attention to the class of theories satisfying $a = c$, and thus possessing a supergravity dual at large N.

At a fixed point, superconformal symmetry ensures that all gauge invariant operators transform with a well-defined weight under a $\mathbb{C}^*$-action, which combines dilatations with R-symmetry rotations. Under a relevant perturbation, the theory flows under the action of dilatations. However, an important insight of Kogan et al. [13] is that in many cases there is a U(1) R-symmetry that remains conserved throughout the flow. This $\mathrm{U}(1)_S$ current is constructed from the geometric R-current R^μ, and the Konishi matter currents K^μ, in such a way that the nontrivial renormalizations of R^μ and K^μ cancel.

To this end recall that the canonical R-current, which lies in the same supermultiplet as the energy-momentum tensor, has a divergence given by

$$\partial_\mu R^\mu = \frac{4}{3}\left[3\mathcal{W} - \sum_i \left(1+\frac{\gamma_i}{2}\right)\Phi_i \frac{\partial \mathcal{W}}{\partial \Phi_i}\right]\Bigg|_{\theta^2} + \frac{1}{48\pi^2}\left[3T(G) - \sum_i T(R_i)(1-\gamma_i)\right] G\tilde{G} + \text{h.c.}\,, \qquad (3)$$

where $\mathcal{W}$ is the superpotential, given in terms of the matter fields Φ_i, which have anomalous dimensions γ_i. We will assume that the superpotential allows for a conserved $\mathrm{U}(1)_R$ current at the classical level. For each matter field, the Konishi current has a one-loop anomaly expressed in the form,

$$\partial_\mu K_i^\mu = 2\left[\Phi_i \frac{\partial \mathcal{W}}{\partial \Phi_i}\right]\bigg|_{\theta^2} + \frac{T(R_i)}{16\pi^2} G\tilde{G} + \text{h.c.} \ . \tag{4}$$

From these relations, one may construct a conserved current via the linear combination [13]

$$S^\mu = R^\mu - \frac{1}{3}\sum_i (\gamma_i - \gamma_i^{\text{IR}}) K_i^\mu \ . \tag{5}$$

In the IR, $S^\mu \to R^\mu$, and thus determines the renormalized R-charges at the fixed point. Moreover, as it is a conserved current, it is not renormalized and its external 't Hooft anomaly is fixed at 1-loop. Thus the R-charges at both UV and IR fixed points can all be determined in the UV from the charges under R^μ and S^μ, since the latter is not renormalized.

This procedure, while straightforward in principle, relies on being able to identify the infrared anomalous dimensions γ_i^{IR}. For chiral operators $\mathcal{O}$, the superconformal algebra relates the dimension to the R-charge,

$$\Delta(\mathcal{O}) = \frac{3}{2} R(\mathcal{O}) \ , \tag{6}$$

and thus it is equivalent to determine the infrared R-charges, r_i^{IR}. The difficulty is that, in general, these charges are unknown. An important constraint is provided by the vanishing of the Adler-Bell-Jackiw (ABJ) anomaly for the R-current. This condition can be written in the form,

$$T(G) + \sum_i T(R_i)(r_i - 1) = 0 \ , \tag{7}$$

which is equivalent (by supersymmetry) to the vanishing of the NSVZ β-function [28]. In certain cases, this condition is sufficient to uniquely determine these charges. In particular, this determines r_i^{IR} in situations where the matter fields all lie in representations with the same R-charge. An important set of examples in this class are $\mathcal{N}=1$ gauge theories with gauge group $\mathrm{SU}(N)$ and N_f fundamental flavors, with N_f lying in the conformal window determined by Seiberg [29].

In general, however, the superconformal R-symmetry corresponds to a specific linear combination of non-anomalous U(1) symmetries and is not fixed uniquely by (7). Recently, Intriligator and Wecht [15] have shown that

Figure 1. Since the background R-current and the energy momentum tensor lie in the same multiplet, the corresponding $\mathrm{Tr}(R^2 J_i)$ and $\mathrm{Tr}(J_i)$ anomalies are related by supersymmetry [15].

in fact this particular linear combination is nonetheless fixed by supersymmetry. Ignoring cases where additional accidental symmetries arise leading to the decoupling of certain composite fields, the constraints are

$$9\mathrm{Tr}(R^2 J_i) = \mathrm{Tr}(J_i) \qquad \text{and} \qquad \mathrm{Tr}(RJ_iJ_j) < 0 \ , \tag{8}$$

where R is the superconformal nonanomalous R-symmetry, and J_i are the remaining nonanomalous U(1) flavor currents. The first of these constraints may be understood on noting that supersymmetry implies that we can turn on a background supermultiplet of external fields containing a source for both the R-current and the energy momentum tensor. The anomalous triangle diagrams related by the first constraint in (8) are then seen to be related by the action of supersymmetry on this multiplet of background fields (see Fig. 1). The second relation in (8) follows from unitarity of the two-point function $\langle J_i J_j \rangle$ and in fact was first obtained in [26]. In particular, $\mathrm{Tr}(RJ_iJ_j)$ is related by supersymmetry to a scale variation of the two-point function $\langle J_i J_j \rangle$ and thus inherits a unitarity constraint which enforces the bound in (8).

We will now interpret $\mathrm{U}(1)_S$ as the specific nonanomalous $\mathrm{U}(1)_R$ current determined by these conditions, as consistent with (5). In addition, superconformal symmetry also relates the R-charges to the central charges in the Weyl anomaly, and consequently the existence of $\mathrm{U}(1)_S$ can be used to determine a and c at the IR fixed point. This procedure was carried out by Anselmi et al. [26, 27] for a large class of models in the $\mathcal{N}=1$ conformal window where the R-charges are uniquely determined by (7). For the class of theories with $a = c$, the results simplify as one has the additional constraint that

$$\mathrm{Tr}(R) = 0 \implies \mathrm{dim}G + \sum_i (\mathrm{dim}R_i)(r_i - 1) = 0 \ , \tag{9}$$

where $\mathrm{dim}G$ is the dimension of the gauge group and $\dim R_i$ is the dimension of the matter field representation. Superconformal symmetry determines the central charges as follows in terms of the R-charges,

$$a = c = \frac{9}{32}\,\mathrm{Tr}(R^3)\ . \tag{10}$$

Intriguingly, Intriligator and Wecht have noted that the conditions (8) are equivalent to finding the local maximum of this cubic function with R given by

$$R_t = R_0 + \sum_i c_i J_i\ , \tag{11}$$

where R_0 is any candidate R-current and c_i are constants to be determined via maximization.

In the $a = c$ class, the R-current anomaly takes the form

$$\partial_\mu \langle \sqrt{g} R^\mu \rangle_{g,F_R} = \frac{2a}{9\pi^2}\, F_{R\mu\nu} \tilde{F}_R^{\mu\nu}\ , \tag{12}$$

where F_R^μ is a source for R^μ, and the metric $g_{\mu\nu}$ is a source for the energy-momentum tensor. It follows that knowledge of the R-charge r_i for each chiral matter field can be translated directly into a result for the central charge $a = c$,

$$a = c = \frac{9}{32} \sum_i (\mathrm{dim} R_i)[(r_i - 1)^3 - (r_i - 1)]\,. \tag{13}$$

Note that the dependence on the massless fermions from the gauge sector has been removed through imposing the $a = c$ constraint. Using the fact that $r_i(UV) = 2/3$ for a free theory, we can then write the ratio of central charges at the IR and UV fixed points as

$$\frac{a_{\mathrm{IR}}}{a_{\mathrm{UV}}} = \frac{27}{8}\, \frac{\sum_i^{\mathrm{UV}} (\mathrm{dim} R_i)[(s_i - 1)^3 - (s_i - 1)]}{\sum_i^{\mathrm{UV}} (\mathrm{dim} R_i)}\ , \tag{14}$$

where the S-charges of fields that have been integrated out necessarily vanishes in the UV, so the sum can always be extended to the entire set of fields. This expression is generic for theories with field content satisfying $a = c$, and we see it depends only on the number of chiral matter fields, and their S-charges. Alternatively, from (5), we need know only those fields which remain massless, and their anomalous dimensions. The restriction that these theories remain within the $a = c$ class is in itself a nontrivial constraint which enforces renormalization of the R-charges as fields are integrated out.

3. Dual hypersurface geometry and R-charges

The dual description of SCFTs that we will consider corresponds to IIB string theory on the near-horizon geometry of a stack of N D3-branes transverse to a space C_6 with a Calabi–Yau metric. In practice, we are interested in placing the D-branes at a singular point in C_6, where a local model of the region near the singularity will be sufficient.

3.1. *Identifying the superconformal R-symmetry*

The AdS/CFT correspondence demands that we can isolate a radial coordinate associated with the distance from the singularity, so that the Calabi–Yau metric on C_6 is conical

$$ds_C^2 = dr^2 + r^2 ds_H^2 \ , \tag{15}$$

where H_5 is the base of the cone. The existence criteria for a Calabi–Yau metric on this cone depend on the precise nature of the singularity at $r = 0$. A generic class on which we will focus involves hypersurface singularities. We consider the hypersurface in $\mathbb{C}^{n+1}$, defined by the vanishing locus of a polynomial $F(z_0, \ldots, z_n)$, such that the hypersurface $F = 0$ is smooth except for an isolated singularity at $z_0 = \cdots = z_n = 0$.

From the condition that there exist a dual SCFT we deduce that the hypersurface must admit a weighted homogeneous $\mathbb{C}^*$-action,

$$F(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d F(z_0, \ldots, z_n) \ , \tag{16}$$

where $w_i \in \mathbb{Z}^+$, so that F defines a weighted projective variety in $\mathbb{WP}_{w_0,\ldots,w_n}$. Writing $\lambda = \mu e^{i\theta}$, we identify the action of μ with dilatations, which ensures that the geometry is conical as in (15), and that of $e^{i\theta}$ with the R-symmetry. To verify the latter correspondence, following [30], we note that the holomorphic n-form,

$$\Omega = \frac{dz_0 \wedge \cdots \wedge dz_n}{dF} \ , \tag{17}$$

has charge

$$w_\Omega = \sum_i w_i - d \tag{18}$$

under $e^{i\theta}$, which is therefore an R-symmetry provided $w_\Omega \neq 0$. This follows from the fact that, for such hypersurfaces, Ω can be written as a quadratic form in covariantly constant spinors on C_6, and thus the spacetime supercharges carry weight $\pm\omega_\Omega/2$. Indeed, we will normalize the charges via the convention that the superspace coordinates should have R-charge $[\theta] = 1$, and thus we require

$$w_\Omega = 2\,. \tag{19}$$

In this case, the existence of a Calabi–Yau metric of the conical form (15) follows from an extension of a theorem of Tian and Yau, discussed in [30].

The existence of the $\mathbb{C}^*$-action, required by superconformal symmetry, then allows us to describe C_6 with its Calabi–Yau metric as a 'complex' cone over a Kähler–Einstein space B_4. The base of the cone H_5 is then Einstein–Sasaki, and can be described as a U(1) fibration over B_4, where we identify the U(1) action on the fibre with a (suitably normalized) R-symmetry. Once the appropriate U(1) isometry is identified, the criterion $\omega_\Omega = 2$ on the scaling of the holomorphic 3-form provides the appropriate normalization condition. However, the identification of the appropriate current is not necessarily straightforward and is a question we will return to below.

With the geometry in hand, it is clear from our discussion in the preceding section that a crucial role will be played by the charges of the coordinates under the fibrewise U(1) isometry. Starting from a UV SCFT, with dual conical background $C_6^{\rm UV}$, under suitable relevant perturbations we expect the field theory to flow to an IR fixed point, and the corresponding geometry to flow to a new conical transverse space $C_6^{\rm IR}$. If the entire RG trajectory admits a dual supergravity description then the existence of the conserved $\mathrm{U}(1)_S$ current requires the existence of a U(1) isometry of C_6 throughout the flow, and not just at the fixed points. The charges of the remaining fields at the IR fixed point will then determine the renormalization of the central charges.

The link between this geometry and the field theory of the previous section is that a single D3-brane will realize the geometry of C_6 through its Higgs-branch moduli space. This relates the coordinate description of the hypersurface above to a set of gauge invariant monomials, which are the natural coordinates within field theory. Thus, within a given model one can relate the weights w_i of the coordinates to the R-charges of the fields in the dual SCFT. In practice, knowledge of the precise map is not vital in calculating the ratio of central charges, since one simply sums up the charges of the chiral fields, and the particular choice of coordinates on the Higgs branch is not important for this.

More problematic is the fact that there may exist several additional non-anomalous $\mathbb{C}^*$ scaling symmetries of the relation $F = 0$ defining C_6, and the precise identification of the $\mathrm{U}(1)_R$ which enters the superconformal algebra may not be straightforward. This will be clear in the example below where the UV fixed point has $\mathcal{N}=2$ supersymmetry and thus an $\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$ global R-symmetry, where the superconformal R-charges arise from mixing between $\mathrm{U}(1)_R$ and a $\mathrm{U}(1)\subset\mathrm{SU}(2)$. The naive non-uniqueness of R-currents, until recently, also hampered the field-theoretic analysis [27]. However, as reviewed above, it has recently become clear that supersymmetry does in fact resolve this apparent puzzle and we will now explore how this is realized on the dual AdS side.

3.2. *Constraints on anomalies from AdS/CFT*

As reviewed above, Intriligator and Wecht have recently shown how the multiplet structure of 't Hooft anomalies in SCFTs is sufficiently strong to determine which nonanomalous U(1) R-current enters the superconformal algebra. In this section we will explore how this is manifest on the dual AdS side within the specific $a = c$ class.

We first need to consider what additional constraints one has on possible 't Hooft anomalies in this case. To this end, it is useful to recall, in analogy with the original matching argument of 't Hooft [31], that if the anomaly is canceled by the addition of spectator fields, then the corresponding current can be gauged. Within string theory, all global symmetries are automatically gauged and one obtains global symmetries only in the limit that certain gauge couplings are sent to zero. At first sight this tells us that all 't Hooft anomalies should vanish, since the corresponding currents can be gauged without the addition of any spectator fields and gauge symmetries cannot be anomalous. One may alternatively view this as the statement that such anomalies are canceled via the Green–Schwarz mechanism [32], namely the required spectator fields, with appropriate Chern–Simons couplings, are always present.

This claim is clearly too strong for the appropriate interpretation of the field theory limit as then all central charges would vanish. The resolution of this puzzle was provided by Witten [3]. The point is that after compactification, the interactions among the massless modes, i.e. the supergravity fields, on AdS_5 contain terms which are *not* gauge invariant (and here we include gravity as a gauge field) when one takes boundary terms into account on AdS_5. For example, there are vector fields gauging the $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_i$ flavor currents, and the corresponding bulk Lagrangian can fail to be gauge invariant due to the presence of Chern–Simons terms. Similarly, and presumably related to it by supersymmetry, the invariance under Weyl transformations is violated due to the presence of a specific log-divergent term in the gravitational action, when evaluated on an asymptotically AdS space. 5D diffeomorphisms in the bulk decompose asymptotically into 4D diffeomorphisms tangential to the boundary, and 4D Weyl transformations orthogonal to it. In this specific background it turns out that the latter are anomalous, reproducing the Weyl anomaly in (1) for $a = c$ [3, 6].

These subtleties do not however affect the $\mathrm{U}(1)_i$ flavor currents, since these symmetries can be gauged in the low energy sector by adding additional branes wrapping cycles in the transverse geometry. Thus we learn that in the presence of background sources for all the relevant currents,

$$\partial_\mu \langle \sqrt{g} J_i^\mu \rangle_{g,F_R,F_i} = 0 \ , \tag{20}$$

i.e. the current is free of 't Hooft anomalies,

$$\mathrm{Tr}(J_i J_j J_k) = 0 \, , \qquad \mathrm{Tr}(J_i R^2) = \mathrm{Tr}(J_i) = 0 \, . \tag{21}$$

Recall from the previous section that the latter two U(1) anomalies are related by supersymmetry [15]. Here this relation is actually visible geometrically since the vector field gauging the R-current corresponds a vector fluctuation of the metric along the S^1 realizing the geometric $\mathrm{U}(1)_R$ isometry [33, 34]. Thus the two anomalies reflect different components of the pure U(1) gravitational anomaly in the full 10D metric background. This vanishes due to the incompatibility of an anomaly with the allowed gauging of $\mathrm{U}(1)_i$ and the residual diffeomorphism invariance in this particular background.

As an aside, we note that in regard to the a-maximization criterion [15] discussed above in reference to (11), the constraint $\mathrm{Tr}(J_i J_j J_k) = 0$ implies that $a(R_t)$ now reduces to a quadratic relation, and thus one requires the *global* maximum. This in turn is given by a linear relation and so the superconformal R-charges will be *rational* numbers in this case.

In contrast to J_i, the R-current does exhibit 't Hooft anomalies due to the presence of Chern–Simons terms as alluded to above. However, there is no purely gravitational anomaly, $\mathrm{Tr}(R)$=0, since one can again turn on the coupling to gravity in the low energy sector. This is equivalent to the restriction $a = c$. Thus the full set of 't Hooft anomalies for the R-current can be expressed in the form

$$\partial_\mu \langle \sqrt{g} R^\mu \rangle_{g,F_R,F_i} = \frac{2a}{9\pi^2} F_{R\mu\nu} \tilde{F}_R^{\mu\nu} - g^{ij} F_{i\mu\nu} \tilde{F}_j^{\mu\nu}, \tag{22}$$

where we have identified the coefficient of the $\mathrm{U}(1)_R^3$ anomaly with the central charge via supersymmetry.

The second term in (22) arises from a set of mixed Chern–Simons terms for bulk vector fields which gauge the U(1) flavor symmetries. The bulk action for these fields contains the terms,

$$\int_{AdS_5} \left[h^{ij} F_i \wedge *F_j - g^{ij} A_R \wedge F_i \wedge F_j \right] . \tag{23}$$

The coupling matrix h^{ij} must have positive-definite eigenvalues for unitarity. Although we will not verify this in detail, it seems clear that $g^{ij} \propto h^{ij}$, which will imply a negativity condition for the anomaly in (22). This relation follows rather directly on noting the 10D origin of these terms, and recalling that A_R is just the graviphoton resulting from fluctuations of S^1_R. The symmetric matrices h^{ij} and g^{ij} then arise from integration over the same geometric cycles, and thus are not independent.

We observe that the positivity of g^{ij} in (22) is entirely consistent with the corresponding field theoretic constraint (8). Indeed, although we have obtained it somewhat differently, it is clear that this relation arises for similar reasons, namely that unitarity requires positivity of the kinetic terms for the gauge fields dual to conserved currents.

4. RG flow to the generalized A_k conifold

In the preceding section we argued that, in a certain class of dual supergravity backgrounds, the R-charges of fields contributing to the central charge could be deduced directly from polynomial relations describing the complex structure (but not the Kähler structure) of the dual background. In this section, as an example of this approach, we consider the class of flows between the specific SCFTs described in Section 1.

4.1. *R-charges from the complex structure*

The UV fixed point we consider is given by an $\mathcal{N}=2$ SCFT associated with N D3-branes transverse to an orbifold $\mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of SU(2) – we will focus on the case $\Gamma = \mathbb{Z}_{k+1}$. This hypersurface is described by the following polynomial equation,

$$F_{\rm UV} = \prod_{i=0}^{k} (x - x_i) + y^2 + z^2 = 0 \ , \tag{24}$$

with moduli x_i as an embedding in $\mathbb{C}^3$ with coordinates (x, y, z). An additional complex transverse coordinate to the D-branes, ϕ, is unconstrained.

The orbifold $F_{\rm UV} = 0$ in (24) descends to a weighted projective variety provided we can identify a suitable $\mathbb{C}^*$-action. A one-parameter family of rescalings is immediately apparent, and is consistent with the normalization (18) provided we choose the following charges,

$$w[y] \,=\, w[z] \,=\, \alpha\,, \quad w[x] = \frac{2\alpha}{k+1}\,, \quad w[\phi] \,=\, 2 - \frac{2\alpha}{k+1}\,, \tag{25}$$

where α is a real or, as argued for above, a rational parameter. The additional coordinate ϕ is transverse to the hypersurface, and thus we find a one-parameter family of candidate weights satisfying the normalization condition. The origin of this ambiguity is easily understood from the fact that there are two orthogonal U(1) isometries that can mix: the first arises from the natural $\mathbb{C}^*$ action on (24), while the second corresponds to rotations in the ϕ-plane. The latter reduces in the near-horizon limit to a great circle on the S^5, on which the twisted-sector fields live [35]. These two isometries are necessarily present due to $\mathcal{N}{=}2$ supersymmetry, and on the field theory side correspond to the two U(1) subgroups of the full $\mathrm{SU}(2)\times\mathrm{U}(1)$ R-symmetry.

To proceed, we may note that since the ϕ-plane is transverse to the hypersurface $F_{\rm UV} = 0$, it must provide a free field within the worldvolume theory of the D3-branes. This fixes the value $\alpha = 2(k+1)/3$, and thus we obtain the following spectrum of weights,

$$w[y] \,=\, w[z] \,=\, \frac{2}{3}(k+1)\,, \quad w[x] = \frac{4}{3}\,, \quad w[\phi] \,=\, \frac{2}{3}\,. \tag{26}$$

Strictly speaking, the reduction of $F_{\rm UV}$ to a weighted projective variety requires integer weights. We have expressed the results in a form appropriate to the normalization condition, but this $\mathbb{C}^*$-action may be decomposed in terms of two components: $R^\mu = 2R_3^\mu/3 + R_{S^1}^\mu/3$, where R_3^μ corresponds to an integer weighted action on x, y and z alone, taking $\alpha = k+1$, and $R_{S^1}^\mu$ corresponds to rotations in the ϕ-plane. In this form, we may identify R_{S^1} and R_3 respectively with the two U(1) Cartan components of the field-theoretic $\mathrm{SU}(2){\times}\mathrm{U}(1)$ R-symmetry.

A point that will be particularly important in what follows is that although we require $N \gg 1$ for a reliable dual supergravity description, the mere existence of this limit will ensure that the dependence on N will cancel from the ratio of central charges. Thus, in considering the map to the Higgs branch of the gauge theory moduli pace, it is sufficient to consider a single D3-brane. The mapping of (x, y, z, ϕ) to gauge invariant monomials can then be obtained by "solving" the hypersurface constraint (24) in terms of new (unconstrained) complex variables X_1 and X_2 at a convenient point in the moduli space where $x_i = 0$. We will also introduce a further variable Φ and identify $\phi = \Phi$, $y + iz = X_1^{k+1}$, $y - iz = X_2^{k+1}$, and $x = X_1 X_2$ [18]. We then read off the following charges under the candidate $\mathrm{U}(1)_R$,

$$[\Phi]_R \,=\, [X_1]_R \,=\, [X_2]_R \,=\, \frac{2}{3}\,. \tag{27}$$

The massless fermions in these chiral superfields then have R-charge -1/3 as expected at the UV fixed point. Given the full background with N D3 branes, we can identify X_1 and X_2 with the set of $k+1$ bi-fundamentals, and Φ with the $k+1$ adjoint chiral multiplets, but the results for a single D3 brane will be sufficient for what follows.

With this normalization of charges at the UV fixed point clarified, we can now consider a deformation breaking $\mathcal{N}=2$ to $\mathcal{N}=1$, by adding mass terms for the adjoint fields Φ. The corresponding geometric deformation is a blowup of the orbifold [17], so that the ADE space (24) is now nontrivially fibred over the ϕ-plane. The hypersurface is given by [18]

$$F_{\mathrm{IR}} = \prod_{i=0}^{k}(x - x_i\phi) + y^2 + z^2 = 0\ , \tag{28}$$

where the moduli x_i are now determined by the mass parameters

$$x_i = -\sum_{j=1}^{i} m_i\ , \tag{29}$$

where m_i is the (normalized) mass for the i^{th} adjoint field. Note that $\sum_i m_i = 0$ so that there is no deformation from the untwisted sector.

The space $F_{\mathrm{IR}} = 0$ descends to a weighted projective variety due to the existence of a $\mathbb{C}^*$-action, normalized again according to (18), with charges

$$w[x] \,= w[\phi] \,=\, 1\,, \qquad w[y] \,=\, w[z] \,=\, \frac{k+1}{2}\ , \tag{30}$$

which in this case is clearly the unique anomaly-free U(1) symmetry and thus we can identify these charges with those of $\mathrm{U}(1)_S$ and consequently with the IR fixed point. Note that the existence of $\mathrm{U}(1)_S$ is crucial in that we can identify these charges after a small perturbation from the UV fixed point, and do not need to carefully verify the form of (28) in the IR. In other words, knowledge of the complex structure is sufficient to determine the S-charges.

From (28) it is apparent that by rescaling the mass parameters x_i in (29) we can restore the complex structure via a compensating rescaling of ϕ, which is not part of the $\mathbb{C}^*$ action admitted by (28). There are thus two scaling relations that must be accounted for in writing the coordinates in terms of unconstrained parameters. In practice, this is straightforwardly accounted for by going to a point in moduli space where the x_i diverge and then using the second scaling symmetry to set $\phi = 0$. We may then solve the constraint by making the identifications: $y+iz = X_1^{k+1}$, $y-iz = X_2^{k+1}$, and $x = X_1X_2$, in analogy with the identifications for (24). In effect the additional rescaling symmetry implies that the combination $x-x_i\phi$ can be treated as one complex variable. In physical terms, it is natural to interpret the additional rescaling symmetry as implying the decoupling of the corresponding field, and this is indeed the picture one has in field theory, where on sending the masses to infinity one decouples all of the adjoint chiral superfields.

From (30), we then read off the charges for the unconstrained parameters (which in field theory correspond to $k+1$ bi-fundamental multiplets),

$$[X_1]_S \;=\; [X_2]_S \;=\; \frac{1}{2}\;, \tag{31}$$

so that their fermionic components have S-charge $-1/2$.

The ratio of central charges may now be evaluated using (14). Simply counting the charges of the unconstrained chiral variables, we have

$$\frac{a_{\rm IR}}{a_{\rm UV}} = \frac{2(s_i^3 - s_i)}{3(r_i^3 - r_i)} = \frac{27}{32}\;, \tag{32}$$

which is the result determined in [23] by calculating the volume of the horizon manifolds. As noted earlier, we only needed to keep track of the unconstrained parameters describing the complex structure, and the result is necessarily independent of k and N, although the charges themselves will scale as $a \propto (k+1)N^2$. With a more complete identification of the gauge theory variables, this dependence could be reconstructed [18], however the relevant point here is that the ratio of central charges can be determined directly from the geometry without a detailed identification of this type. This conclusion may also be drawn from an inspection of the explicit formula for the volume of the five-dimensional Einstein–Sasaki base manifold [23]. With our normalization condition (19), this volume can be written in the form

$$V(H_5) = \frac{8}{27}\pi^3 \frac{\sum w_i - 2}{\prod w_i}\;, \tag{33}$$

which determines the central charge via (2). We see that this result again depends only on the weights w_i of the $\mathbb{C}^*$-action on the polynomial equation defining the complex structure of the hypersurface.

As a final remark, we note also that formally the result above for $a_{\rm IR}/a_{\rm UV}$ is correct for $k=0$, corresponding to no orbifold action in the UV, which is the flow from $\mathcal{N}=4$ SYM discussed by Gubser et al. [36]. However, in this case the mass perturbation is from an untwisted sector (clearly as there is no orbifold), and such perturbations cannot be described as deformations of the complex structure in the manner illustrated above.

4.2. *Marginal deformations and moduli*

The geometric description of the Higgs branch at the IR fixed point given in (28), makes transparent the presence of a number of marginal perturbations. In particular, the k independent complex mass parameters determine the moduli x_i. If all are nonzero, and one is chosen to set the scale, the other $k-1$ provide homogeneous coordinates on $\mathbb{CP}^{k-1}$, which specifies the part of the moduli space for the fixed point theory which is determined by complex structure deformations of the Higgs branch [25]. One can ask what happens when some of the mass parameters are set to zero. The discussion above then applies only to those sectors with nonzero adjoint mass. However, the couplings to the bi-fundamentals ensures that this propagates to other sectors. In particular, if only one mass term is nonzero it is clear that the couplings will generate anomalous dimensions for all the bi-fundamentals by propagating round the A_k quiver diagram. Thus it seems that the only consistent fixed point will have $a_{\rm IR}/a_{\rm UV}$ as given in (32).

Setting some mass parameters to zero then restricts to certain projective submanifolds in the full moduli space $\mathbb{CP}^{k-1}$. Note that one can also turn on a mass for a diagonal combination of the adjoints, corresponding to an operator in the untwisted sector. This is not apparent in the complex structure of the Higgs branch, but has been discussed in more detail in [25].

There are also moduli associated with the Coulomb branch which are determined by various combinations of the gauge couplings in the UV. The UV theory falls into the class of elliptic models studied by Witten, and the moduli space is that of a torus with k marked points [37].

5. Concluding Remarks

The analysis of the preceding section relied to a certain extent on the realization of these backgrounds as algebraic varieties, and one may wonder whether this structure was strictly necessary for deducing the R-charges. In principle, this should not be the case since the requirement that C_6 admit a $\mathbb{C}^*$-action is present on algebraic grounds whether or not the background admits a global toric description. It would certainly be interesting to see if similar calculations are possible in other cases. A simple example would be the $k = 0$ "limit" of the flows considered here where the perturbation from the UV fixed point is necessarily in the untwisted sector.

The question of whether the $\mathbb{C}^*$-action should be well-defined along the entire flow, rather than just at the fixed points, is more difficult to answer. Firstly, it may not be generic that the entire flow lies in the supergravity regime. Secondly, the "complexification" of the renormalization group-action relies on the validity of the identification of the $\mathrm{U}(1)_S$ current at the UV fixed point. In general, accidental symmetries may arise along the flow which can imply that the R-charges in the IR are not identifiable after a "linear" perturbation from the UV. In such cases, it seems unlikely that within the string dual the IR complex structure would be visible as a simple blowup or deformation of the UV complex structure.

We will finish with a brief remark on the identification of the superconformal R-current. In particular, the precise reason that one can determine the superconformal R-charges via the maximization of a cubic function, namely $a(R_t)$ [15], over the space of nonanomalous U(1) currents still seems rather obscure. In this regard, Kutasov [38] has recently reformulated this construction by introducing a quantity, that we shall call a_S, given by

$$a_S(\eta, \lambda_a, c_i) = a(R_t(c_i)) - \eta\Big[T(G) + \sum_i T(R_i)(r_i - 1)\Big] - \sum_a \lambda_a \left[R(\mathcal{W}_a) - 2\right], \tag{34}$$

to be extremized over a larger space of parameters. In particular, Lagrange multipliers are introduced to enforce the vanishing of the ABJ anomaly for the R-current, and the marginality of any superpotential terms, where $R(\mathcal{W}_a)$ denotes the R-charge of the term $\mathcal{W}_a$. Kutasov has argued that a_S evolves monotonically between UV and IR fixed points as a function of η, provided the evolution is perturbative.

Here we would simply like to point out the similarity of this prescription with the Gibbs maximization principle, whereby one obtains the Gibbs measure $\rho = e^{-\beta H}/Z$ via extremizing the Shannon entropy $S = -\mathrm{Tr}(\rho \ln \rho)$ subject to the constraints defining the canonical ensemble. Of particular interest in regard to questions concerning the irreversibility of RG flow is that, in order to make this analogy, we need to identify $a(R_t)$ with the Shannon entropy! It may be that a consideration of the AdS dual along the lines we have discussed will shed more light on this intriguing, but thus far rather vague, connection.

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