FINITE DENSITY STATES IN INTEGRABLE CONFORMAL FIELD THEORIES

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We study states of large charge density in integrable conformal coset models. For the $O(2)$ coset, we consider two different S-matrices, one corresponding to a Thirring mass perturbation and the other to the continuation to $O(2+\epsilon)$. The former leads to simplification in the conformal limit; the latter gives a more complicated description of the $O(2)$ system, with a large zero mode sector in addition to the right- and left-movers. We argue that for the conformal $O(2+2M|2M)$ supergroup coset, the S-matrix is given by the analog of the $O(2+\epsilon)$ construction.

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1. Introduction

The $AdS_5 \times S^5$ background of IIB string theory is highly symmetric, and one might hope that the string world-sheet theory in this background would be exactly solvable. However, because of the presence of R–R flux, the usual tools such as current algebra are not available. In Ref. [1–4] it was shown that the world-sheet CFT has an infinite number of nonlocal conserved charges of the type that arise in integrable models, at least at the classical level (see Refs. [5–7] for further developments). There has been some discussion of the combination of integrability and conformal invariance [8–10], but thus far there are no methods with the power and generality of current algebra or rational conformal field theory. Thus, for the $AdS_5 \times S^5$ world-sheet theory, and more generally for CFT's based on supermanifold sigma models [11–16], there is no known way to calculate the energies of general world-sheet states.

In this paper we would like to take small steps in this direction. There are well-established methods for calculating the energies of states with large densities of a conserved charge, starting from the exact continuum S-matrix [17–22]. We would like to examine the conformal limit of these calculations, and then apply them to the conformal $OSp(2+2M|2M)$ coset model.

In Sec. 2 we develop the conformal limit of the finite density system. The calculation separates into decoupled right- and left-moving calculations, which are simpler than in the nonconformal case. As a warmup we apply this first to the case $M = 0$, the bosonic $O(2)$ model. In Sec. 3 we use the normal Thirring description of the $O(2)$ model, which has a simple massless limit. In Sec. 4 we consider a different description of the $O(2)$ model, as the $N \to 2$ limit of the $O(N)$ model. This gives a different S-matrix, describing the bosonic spins of the $O(2)$ model rather than the Thirring fermions. The limiting process seems to be sensible but the result is more complicated than previous examples of conformal integrable models, in that there is a large and nontrivial zero-mode sector in addition to the right- and left-movers. In Sec. 5 we argue that the $OSp(2+2M|2M)$ model should be given by the lift of the *second* description of the $O(2)$ model. Section 6 discusses further directions.

Of course, there has been an explosion of work on integrability from the gauge theory side of the AdS/CFT duality, beginning with Refs. [23, 24]. At present, it appears that progress in this direction is much easier than on the string sigma model side. However, it seems likely that a perspective from both sides of the duality will ultimately be useful. We should note that states with large spin have considered extensively on the gauge theory side as well (e.g. [25, 26]; for a review see ref. [27]). Also, there have been efforts to derive the string sigma model directly from the spin chain on the gauge theory side [28–34] and to relate the integrable structures on the two sides [35, 36]; we do not know if there is a connection with our work.

2. Finite Density in the Conformal Limit

We start with a 1+1 dimensional relativistic theory, whose exact S-matrix is assumed to be known. We are interested in the the lowest energy state with a specified charge and momentum, so we consider the case that we only have particles of one type, with a given sign of the charge, and that all states are filled in a range of rapidities $-B_L < \theta < B_R$. We then have the standard Bethe ansatz equation [37, 38],

$$m\cosh\theta + \int_{-B_L}^{B_R} K(\theta-\theta')\rho(\theta')d\theta' = \rho(\theta)\ , \quad -B_L < \theta < B_R\ . \tag{1}$$

Here ρ is 2π times the density of particles per unit length *and* unit rapidity, so that the number density per unit length is given by the rapidity integral

$$\mathcal{J} = \frac{1}{2\pi}\int_{-B_L}^{B_R}\rho(\theta)d\theta\ . \tag{2}$$

The kernel is

$$K(\theta) = \frac{1}{2\pi i}\,\partial_\theta \ln S(\theta)\ , \tag{3}$$

where $S(\theta)$ is the S-matrix between two particles of the given type. Note that the integral equation holds only in the range $-B_L < \theta < B_R$ in which $\rho(\theta)$ is nonzero. The equation is complicated because this range is bounded on both ends. It can be analyzed using the Wiener-Hopf technique [18–20, 22], but in general cannot cannot be solved in closed form.

Now let us take the limit $m \to 0$, holding fixed the momentum. For right- and left-moving particles,

$$\begin{aligned} p_R &= m\sinh\theta \approx \frac{m}{2}\,e^{\theta} = \frac{\mu}{2}\,e^{\tilde\theta_R}\ , \quad \theta = \tilde\theta_R + \ln\frac{\mu}{m}\ , \\ p_L &= m\sinh\theta \approx -\frac{m}{2}\,e^{-\theta} = -\frac{\mu}{2}\,e^{-\tilde\theta_L}\ , \quad \theta = \tilde\theta_L - \ln\frac{\mu}{m}\ , \end{aligned} \tag{4}$$

where μ is a fixed reference scale. Thus we hold fixed $\tilde\theta_{R,L}$ in the limit. We assume that in the massless limit the density separates into a right-moving part which is a function of $\tilde{\theta}_R$ and a left-moving part which is a function of $\tilde{\theta}_L$:

$$\rho_R(\tilde{\theta}_R) = \lim_{m\to 0} \rho(\tilde{\theta}_R + \ln \mu/m) \ ,$$
$$\rho_L(\tilde{\theta}_L) = \lim_{m\to 0} \rho(\tilde{\theta}_L - \ln \mu/m) \ . \tag{5}$$

Since the S-matrix depends only on rapidity differences, the RR and LL S-matrices are the same as the original S-matrix,

$$S_{RR}(\tilde{\theta} - \tilde{\theta}') = S_{LL}(\tilde{\theta} - \tilde{\theta}') = \lim_{m\to 0} S(\tilde{\theta} - \tilde{\theta}') \ . \tag{6}$$

On the other hand, for right- and left-moving particles the rapidity difference is diverging in the limit and so [a]

$$S_{RL}(\tilde{\theta} - \tilde{\theta}') = \lim_{m\to 0} \lim_{\theta\to\infty} S(\theta) \ . \tag{7}$$

The Bethe ansatz equation then separates into two pieces, which are obtained by holding $\tilde{\theta}_R$ or $\tilde{\theta}_L$ fixed as $m \to 0$:

$$\frac{\mu}{2} e^{\tilde{\theta}} + \int_{-\infty}^{\tilde{B}_R} K(\tilde{\theta} - \tilde{\theta}')\rho_R(\tilde{\theta}')d\tilde{\theta}' = \rho_R(\tilde{\theta}) \ , \quad -\infty < \tilde{\theta} < \tilde{B}_R \ ,$$
$$\frac{\mu}{2} e^{-\tilde{\theta}} + \int_{-\tilde{B}_L}^{\infty} K(\tilde{\theta} - \tilde{\theta}')\rho_L(\tilde{\theta}')d\tilde{\theta}' = \rho_L(\tilde{\theta}) \ , \quad -\tilde{B}_L < \tilde{\theta} < \infty \ , \tag{8}$$

where $\tilde{B}_{R,L} = B_{R,L} - \ln \mu/m$ is fixed in the limit. There is no RL cross term because $\partial_\theta S$ vanishes at large rapidity for all cases of interest.

Because the original rapidity range $B_R + B_L$ diverges in the limit, the right- and left-moving rapidity ranges are each bounded on only one side, and these integral equations can be solved in closed form. We follow the Wiener-Hopf technique, as described for example in the appendix to [18] and in [21]. Focussing on the right-moving equation, we write it as

$$g(\tilde{\theta}) - \rho_R(\tilde{\theta}) + \int_{-\infty}^{\tilde{B}_R} K(\tilde{\theta} - \tilde{\theta}')\rho_R(\tilde{\theta}')d\tilde{\theta}' = X(\tilde{\theta}) \ , \tag{9}$$

where $X(\tilde{\theta})$ is nonvanishing only for $\tilde{\theta} > \tilde{B}_R$. Here $g(\tilde{\theta}) = \frac{1}{2}\mu e^{\tilde{\theta}} H(\tilde{B}_R - \tilde{\theta})$, where H denotes the step function. Taking the Fourier transform $\int_{-\infty}^{\infty} d\tilde{\theta}\, e^{i\omega\tilde{\theta}}$

[a] We are assuming here that the limits $m \to 0$ and $\theta \to \infty$ commute. This will be true for the Thirring S-matrix studied in Sec. 3, which is simply independent of m, but it will not be true for the limit in Sec. 4, which will require a more complicated treatment.

on both sides gives

$$\tilde{g}(\omega) - [1 - \tilde{K}(\omega)]\tilde{\rho}(\omega) = \tilde{X}(\omega) \ . \tag{10}$$

Because of the bounded ranges of ρ_R, g, and X, it follows that

$$\begin{aligned} \tilde{\rho}_R(\omega) &= e^{i\omega \tilde{B}_R}\rho_{R-}(\omega) \ , \\ \tilde{g}(\omega) &= e^{i\omega \tilde{B}_R} g_-(\omega) \ , \\ \tilde{X}(\omega) &= e^{i\omega \tilde{B}_R} X_+(\omega) \ , \end{aligned} \tag{11}$$

where the subscripts $\pm$ denote functions which are holomorphic in the upper and lower half-planes respectively. These functions also vanish asymptotically in the half-planes where they are holomorphic, because ρ_R, g and X have finite discontinuities at $\tilde{B}_R$.

Given a bounded function $\Psi(\omega)$ which vanishes at $\omega \to \pm\infty$, we can define

$$\Psi_\pm(\omega) = \pm\frac{1}{2\pi i}\lim_{\delta\to 0}\int_{-\infty}^{\infty}\frac{d\omega'\,\Psi(\omega')}{\omega' - (\omega \pm i\delta)} \ . \tag{12}$$

These are holomorphic in the upper and lower half-plane respectively, and moreover $\Psi(\omega) = \Psi_+(\omega) + \Psi_-(\omega)$. The operations $[\ \ ]_\pm$ act as projection operators, in that $[f_-]_+ = 0$ and $[f_-]_- = f_-$. Applying this to $\ln[1 - \tilde{K}(\omega)]$ (which vanishes asymptotically for smooth $K(\theta)$), it follows that we can write

$$1 - \tilde{K}(\omega) = \frac{1}{G_+(\omega)G_-(\omega)} \ , \tag{13}$$

where $G_\pm(\omega)$ are holomorphic and nonvanishing in the upper and lower half-planes respectively, and approach 1 asymptotically. The integral equation can thus be put in the form

$$\frac{\rho_{R-}(\omega)}{G_-(\omega)} = G_+ g_-(\omega) - G_+(\omega)X_+(\omega) \ . \tag{14}$$

Taking the $[\ \ ]_-$ part (12) eliminates the unknown function $X_+(\omega)$,[b] giving

$$\frac{\rho_{R-}(\omega)}{G_-(\omega)} = [G_+ g_-(\omega)]_- \ . \tag{15}$$

[b] This is the point where the simplification due to a semi-infinite range enters. Otherwise there would be a second unknown function $X_-(\omega)$, and an additional step would be needed, leading to an integral equation that cannot be solved in closed form.

Finally, using the explicit form $g_-(\omega) = \mu e^{\tilde{B}_R}/2(1+i\omega)$ allows us to evaluate the contour integral explicitly, giving

$$\tilde{\rho}_R(\omega) = \frac{\mu\, e^{(1+i\omega)\tilde{B}_R} G_+(i) G_-(\omega)}{2(1+i\omega)} \ . \tag{16}$$

The rapidity density $\rho_R(\theta)$ is obtained from the inverse Fourier transform, but the quantities of main interest are given directly by $\tilde{\rho}_R(\omega)$. The total charge density carried by the right-movers is

$$\begin{aligned} \mathcal{J}_R = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_R(\tilde{\theta})\, d\tilde{\theta} &= \frac{1}{2\pi}\, \tilde{\rho}_R(0) \\ &= \frac{\mu e^{\tilde{B}_R}}{4\pi}\, G_+(i) G_-(0) \ . \end{aligned} \tag{17}$$

This determines $\tilde{B}_R$ in terms of $\mathcal{J}_R$. The total energy and momentum densities are

$$\begin{aligned} \mathcal{E} = \mathcal{P} = \frac{\mu}{4\pi} \int_{-\infty}^{\infty} e^{\tilde{\theta}} \rho_R(\tilde{\theta})\, d\tilde{\theta} &= \frac{\mu}{4\pi}\, \tilde{\rho}_R(-i) \\ &= \frac{\mu^2 e^{2\tilde{B}_R}}{16\pi}\, G_+(i) G_-(-i) \ . \end{aligned} \tag{18}$$

In general $K(\theta) = K(-\theta)$, and so $G_+(i) = G_-(-i)$. Then we can write

$$\mathcal{E} = \mathcal{P} = \frac{\pi \mathcal{J}_R^2}{G_+(0) G_-(0)} = [1 - \tilde{K}(0)] \pi \mathcal{J}_R^2 \ . \tag{19}$$

Similarly, for left-movers

$$\mathcal{E} = -\mathcal{P} = [1 - \tilde{K}(0)] \pi \mathcal{J}_L^2 \ . \tag{20}$$

The relation between the energy and charge thus depends only on the total change in the phase of S from $\theta = -\infty$ to $\theta = \infty$.

Note that we have discussed only densities in a system of infinite volume. In a finite volume system there will be corrections, Casimir effects. Obtaining these from the infinite volume S-matrix which is our starting point is a difficult problem for which there is only a partial solution; we will comment on this further in the conclusions. For the bulk of this paper we focus on the infinite volume case, or equivalently on the *leading* high-density properties in a system of finite volume.

3. The Massless Thirring Model

The fermionic and bosonic descriptions of the Thirring model are

$$S = \int d^2x \left[i\bar{\psi}\gamma^\mu \partial_\mu \psi + \frac{\lambda}{2}(\bar{\psi}\psi)^2 - m\bar{\psi}\psi \right] \tag{21}$$

and

$$S = -\int d^2x \left[\frac{1}{2g^2}\partial_\mu \phi \partial^\mu \phi + m\cos\phi \right] . \tag{22}$$

The field ϕ is normalized to have periodicity 2π, so that a fermion corresponds to a kink $\Delta\phi = 2\pi$. The Thirring fermion-fermion S-matrix is [39]

$$S = \frac{\Gamma\Big(\frac{8\pi}{\gamma}\Big)\Gamma\Big(1 + \frac{8i\theta}{\gamma}\Big)}{\Gamma\Big(\frac{8\pi}{\gamma} + \frac{8i\theta}{\gamma}\Big)} \prod_{n=1}^{\infty} \frac{R_n(\theta)R_n(i\pi - \theta)}{R_n(0)R_n(i\pi)} \tag{23}$$

where

$$R_n(\theta) = \frac{\Gamma\Big(2n\frac{8\pi}{\gamma} + \frac{8i\theta}{\gamma}\Big)\Gamma\Big(1 + 2n\frac{8\pi}{\gamma} + \frac{8i\theta}{\gamma}\Big)}{\Gamma\Big([2n+1]\frac{8\pi}{\gamma} + \frac{8i\theta}{\gamma}\Big)\Gamma\Big(1 + [2n-1]\frac{8\pi}{\gamma} + \frac{8i\theta}{\gamma}\Big)} . \tag{24}$$

Only when the Thirring mass is nonzero can this be interpreted as an S-matrix in the usual sense, but even in the massless limit it can be used sensibly in the Bethe ansatz [8]. The S-matrix contains a dimensionless parameter γ which is related to the couplings in the fermionic and bosonic description by [39]

$$\frac{8\pi}{\gamma} = 1 - \frac{\lambda}{\pi} = \frac{8\pi}{g^2} - 1 . \tag{25}$$

In particular, $\gamma = 8\pi$ and $g^2 = 4\pi$ is the free fermion theory.

The Fourier transform of the kernel is fairly simple,

$$\tilde{K}(\omega) = \frac{\sinh\Big(\frac{\gamma\omega}{16} - \frac{\pi\omega}{2}\Big)}{2\sinh\frac{\gamma\omega}{16}\cosh\frac{\pi\omega}{2}} . \tag{26}$$

Thus

$$1 - \tilde{K}(0) = \frac{1}{2}\left(1 + \frac{8\pi}{\gamma}\right) = \frac{4\pi}{g^2} , \tag{27}$$

and so the energy and momentum densities are

$$\mathcal{E} + \mathcal{P} = \frac{8\pi^2}{g^2}\,\mathcal{J}_R^2 , \qquad \mathcal{E} - \mathcal{P} = \frac{8\pi^2}{g^2}\,\mathcal{J}_L^2 . \tag{28}$$

On the other hand, canonical quantization gives

$$\mathcal{E} \pm \mathcal{P} = \frac{1}{2g^2} (\phi' \pm g^2 \Pi)^2 \ , \qquad \Pi = \dot{\phi}/g^2 \tag{29}$$

in the bosonic description of the massless theory.

There is an obvious correspondence between the integrable and canonical results (28) and (29). However, it is slightly subtle to understand directly the relation between the quantum numbers $\mathcal{J}_R, \mathcal{J}_L$ of the integrable description and those of the canonical description. In the latter, there are two conserved charge densities. The topological charge density $\frac{1}{2\pi}\phi'$ is the total fermion number density; this follows from our normalization of the field ϕ to periodicity 2π. Thus we identify $\mathcal{J}_R + \mathcal{J}_L = \frac{1}{2\pi}\phi'$. The Noether density 2Π is the chiral fermion number density; the normalization follows from the fact that $e^{\pm i\phi}$ are fermion bilinears. We will see that this quantum number is more subtle to identify in the integrable case. The conserved fermionic charges are

$$\mathcal{N}_{R,L} = \frac{1}{2} \left(\frac{1}{2\pi} \phi' \pm 2\Pi \right) . \tag{30}$$

We can now make two quick checks. For the parity-symmetric state $\mathcal{J}_R = \mathcal{J}_L = \frac{1}{4\pi}\phi'$, the chiral density Π vanishes and in this case the energies (28) and (29) agree for all g. For the free-fermion case $g^2 = 4\pi$, the fermions in the integrable and canonical descriptions are the same,

$$\mathcal{J}_{R,L} = \mathcal{N}_{R,L} \qquad (g^2 = 4\pi) \ , \tag{31}$$

and with this the energies (28) and (29) match at the free fermion point.

To match the quantum numbers in general, let us start at the free fermion point and consider an adiabatic variation of g. The densities (30) are constructed from the topological and Noether densities without g-dependence, and so are invariant. In terms of these, the canonical energy/momentum density (29) is

$$\mathcal{E} \pm \mathcal{P} = \frac{2\pi^2}{g^2} \left[(\mathcal{N}_R + \mathcal{N}_L) \pm \frac{g^2}{4\pi} (\mathcal{N}_R - \mathcal{N}_L) \right]^2 . \tag{32}$$

To follow the quantum numbers in the integrable case, let us back up one step to the 'undifferentiated' Bethe ansatz

$$m \sinh\theta + \frac{1}{2\pi i} \int_{-B_L}^{B_R} \ln S(\theta - \theta')\rho(\theta')d\theta' = \frac{2\pi n}{L} \ , \quad -B_L < \theta < B_R \ . \tag{33}$$

We have introduced a finite volume L; n is an integer labeling the particle states. Eq. (1) is obtained from this by taking the difference for consecutive values of n. All states in the range $-n_L < n < n_R$ are filled; the rapidity endpoints $B_{R,L}$ are implicitly determined in terms of $n_{R,L}$. At the free fermion point we can immediately identify the number densities

$$\mathcal{N}_{R,L} = \frac{n_{R,L}}{L} \ . \tag{34}$$

Both $\mathcal{N}_{R,L}$ and $n_{R,L}$ are adiabatically invariant, so this holds for all g. Consider Eq. (33) in the conformal limit, taking $\tilde{\theta}_R \to -\infty$:

$$\begin{aligned} \frac{1}{2\pi i}\int_{-\tilde{B}_L}^{\infty} \ln S_{LR}\rho_L(\tilde{\theta}')d\tilde{\theta}' + \frac{1}{2\pi i}\int_{-\infty}^{\tilde{B}_R} \ln S_{RR}(-\infty)\rho_R(\tilde{\theta}')d\theta' = \frac{2\pi}{L}\, n_{\tilde{\theta}_R\to-\infty} \\ \frac{1}{2\pi i}\int_{-\tilde{B}_L}^{\infty} \ln S_{LL}(\infty)\rho_L(\tilde{\theta}')d\tilde{\theta}' + \frac{1}{2\pi i}\int_{-\infty}^{\tilde{B}_R} \ln S_{RL}\rho_R(\tilde{\theta}')d\theta' = \frac{2\pi}{L}\, n_{\tilde{\theta}_L\to\infty} \ . \end{aligned} \tag{35}$$

This determines the density $\mathcal{J}_R$, because the total number of filled right-moving states is

$$L\mathcal{J}_R = n_R - n_{\tilde{\theta}_R=-\infty} \ . \tag{36}$$

Noting that

$$\frac{1}{2\pi i}\ln S_{LR} = -\frac{1}{2\pi i}\ln S_{RR}(-\infty) = \frac{1}{2}\,\tilde{K}(0) = \frac{1}{2}\left[1 - \frac{4\pi}{g^2}\right] , \tag{37}$$

the integrals (35) just involve the total densities, giving

$$\frac{1}{2}\left[1 - \frac{4\pi}{g^2}\right](\mathcal{J}_L - \mathcal{J}_R) = \mathcal{N}_R - \mathcal{J}_R = \mathcal{J}_L - \mathcal{N}_L \ , \tag{38}$$

where the last equality follows from a similar calculation for the left-movers. Then

$$\mathcal{J}_{R,L} = \frac{1}{2}\left[(\mathcal{N}_R + \mathcal{N}_L) \pm \frac{g^2}{4\pi}\,(\mathcal{N}_R - \mathcal{N}_L)\right] . \tag{39}$$

With this identification of quantum numbers the energy/momentum densities (28) calculated using the integrable description do indeed reduce to those (32) obtained in the canonical description.

Again, the canonical currents (30) are conserved under adiabatic variation of g, but the currents $\mathcal{J}_{R,L}$ are anomalous. The Bethe equation (33) determines θ as a function of n and g. As we vary g at fixed n, states move to the lower end of the right-moving spectrum and reappear on the upper end of the left-moving spectrum — there is a spectral flow.

So far we have focused on the case that only a band of particle states is filled, so that $\mathcal{J}_R$ and $\mathcal{J}_L$ are positive. However, it is clear from Eq. (39) that as we vary g one of these, say $\mathcal{J}_L$, may go to zero. What happens next is different in the nonconformal theory, for arbitrarily small m, than in the conformal theory of interest: the massless limit does not commute with adiabatic evolution. In the massive theory, when a left-moving particle state passes through zero rapidity it becomes a right-moving particle state, so that after the last left-moving particle has passed through zero we end up with a bounded interval of filled positive rapidity particle states. In the massless case, when a left-moving particle state passes through zero it become an empty left-moving antiparticle state. After the last filled left-moving state passes through zero, empty left-moving particle states pass through to become filled left-moving antiparticle states. Thus, negative values of $\mathcal{J}_{R,L}$ and $\rho_{R,L}$ signify filled antiparticle states.

The derivation of the conformal Bethe ansatz equations (8) assumed particle states, but in fact these equations continue to hold. The particle-antiparticle reflection amplitude S_R goes to zero at large rapidity and the particle-antiparticle transmission amplitude S_T goes to a constant [39], so the right- and left-moving equations continue to decouple. Thus these equations apply for any signs of $\mathcal{J}_R$ and $\mathcal{J}_L$, as long as all right-movers are of the same type, and similarly all left-movers.

4. The $N \to 2$ Limit of the $O(N)$ Coset Model

In the classic study of $O(N)$-invariant S-matrices [39], the case $N = 2$ required a separate treatment from $N > 2$. For example, the minimal $O(2)$ S-matrix (23) contains the free parameter γ, while there is no free parameter for $N > 2$. Thus the $N = 2$ S-matrix cannot be thought of as a limit from $N > 2$. However, we will argue in the next section that in order to treat the supergroup coset we need the analog of the $N \to 2$ limit of the S-matrix of the $O(N)$ sigma model. We can think of this as corresponding to a different massive perturbation of the conformally invariant $O(2)$ model, turning on a nonzero β-function at $N = 2+\epsilon$ rather than a nonzero fermion mass as in the Thirring description. Of course, it is not clear a priori that this procedure is physically sensible, but we will try it and see. We find that the $N \to 2$ limit of the Bethe ansatz appears to exist, but that it is more complicated than the conformal limits encountered thus far.

The sigma model action is

$$S = -\frac{1}{2g^2}\int d^2x\,\partial_\mu\varphi^i\partial^\mu\varphi^i\ , \quad \varphi^i\varphi^i = 1\ , \quad i = 1,\ldots,N\ . \tag{40}$$

For $N = 2$, $\varphi^1 + i\varphi^2 = e^{i\phi}$ gives the bosonic action (22) at $m = 0$. The coupling g runs for $N > 2$ but this running turns off in the limit, so by appropriately scaling the energy as we take $N \to 2$ we can obtain different fixed values of g. The β-function is

$$\mu\,\frac{\partial g}{\partial\mu} = (N-2)F(g)\ , \quad F(g) = -\frac{g^3}{4\pi} - \frac{g^5}{8\pi^2} + \ldots\ . \tag{41}$$

The coupling thus runs at a rate proportional to $N - 2$,

$$(N-2)\ln\frac{\mu}{m} = \frac{2\pi}{g^2} + \ln g^2 + \text{const.} + \ldots \equiv \chi(g)\ , \tag{42}$$

where m is the dynamically generated mass scale. Identifying $\mu \sim E \sim me^{|\theta|}$, we see that when we hold E and g fixed as $N \to 2$, the dynamical mass m goes to zero, and also we must hold fixed $|\theta| - \chi(g)/(N-2)$. That is, we focus on a rapidity region where the coupling takes a specified value g in the limit.

The S-matrix for the $O(N)$ sigma model decomposes into three terms

$$|k\,\theta, l\,\theta'; \text{in}\rangle = S_{kl,ij}(\theta-\theta')|i\,\theta, j\,\theta'; \text{out}\rangle\ ,$$
$$S_{kl,ij}(\theta) = \delta_{ij}\delta_{kl}\sigma_1^+(\theta) + \delta_{ik}\delta_{jl}\sigma_2^+(\theta) + \delta_{il}\delta_{jk}\sigma_3^+(\theta)\ , \tag{43}$$

where $\sigma^+_{1,2,3}(\theta-\theta')$ are given in Ref. [39]. As in the limiting process of Sec. 1, RR and LL scattering involve finite differences in rapidity while RL scattering involves rapidity differences that diverge in the limit, as $1/(N-2)$. The $O(N)$ sigma model S-matrix for same-charge scattering is $S = \sigma_2^+ + \sigma_3^+$, which is

$$S(\theta) = \frac{\Gamma\Big(\frac{1}{N-2} - \frac{i\theta}{2\pi}\Big)\Gamma\Big(\frac{1}{N-2} + \frac{1}{2} + \frac{i\theta}{2\pi}\Big)\Gamma\Big(\frac{1}{2} - \frac{i\theta}{2\pi}\Big)\Gamma\Big(\frac{i\theta}{2\pi}\Big)}{\Gamma\Big(\frac{1}{N-2} + \frac{i\theta}{2\pi}\Big)\Gamma\Big(\frac{1}{N-2} + \frac{1}{2} - \frac{i\theta}{2\pi}\Big)\Gamma\Big(\frac{1}{2} + \frac{i\theta}{2\pi}\Big)\Gamma\Big(\frac{-i\theta}{2\pi}\Big)}\ . \tag{44}$$

The limit relevant to the LL and RR S-matrices is taken with fixed rapidity,

$$\lim_{N\to 2} S(\theta) = \frac{\Gamma\Big(\frac{1}{2} - \frac{i\theta}{2\pi}\Big)\Gamma\Big(\frac{i\theta}{2\pi}\Big)}{\Gamma\Big(\frac{1}{2} + \frac{i\theta}{2\pi}\Big)\Gamma\Big(\frac{-i\theta}{2\pi}\Big)} \equiv S_{\rm I}(\theta)\ . \tag{45}$$

The limit relevant to the LR S-matrix is taken with rapidity proportional to $1/(N-2)$, because the right- and left-movers are localized near $\theta =$

$\pm\chi(g)/(N-2)$. Thus we define

$$\lim_{N\to 2} S\big(\zeta/[N-2]\big) = \left(\frac{2\pi + i\zeta}{2\pi - i\zeta}\right)^{1/2} e^{i\pi\,\mathrm{sign}(\zeta)/2} \equiv S_{\mathrm{II}}(\zeta)\ , \tag{46}$$

and $S_{LR} = S_{\mathrm{II}}(2\chi(g))$. The coupling does not appear in S_{LL} and S_{RR}, but does appear in S_{LR}.

Now, however, we encounter an interesting complication. In the Thirring model we had

$$S_{LR} = S_{LL}(\tilde\theta \to \infty)\ , \quad S_{RL} = S_{RR}(\tilde\theta \to -\infty)\ , \tag{47}$$

so the particle numbers in Eq. (35) satisfy $n_{\tilde\theta_R \to -\infty} = n_{\tilde\theta_L \to \infty}$. That is, there are no missing n's between the right- and left-movers. In the present case, this cannot hold in general, because S_{RR} and S_{LL} do not depend on the coupling while S_{RL} and S_{LR} do. Thus, there is a range of n that correspond to what we will call 'zero mode' states, in the large rapidity regime between the right- and left-movers. If we solve the Bethe ansatz for $N > 2$ and then take the limit, the rapidity distribution must approach such a form. Thus we generalize the earlier conformal limit (5) to [c]

$$\begin{aligned}
\rho_R(\tilde\theta_R) &= \lim_{N\to 2} \rho\big(\tilde\theta_R + \chi(g)/[N-2]\big)\ , \quad -\infty < \tilde\theta_R < \tilde B_R\ ,\\
\rho_L(\tilde\theta_L) &= \lim_{N\to 2} \rho\big(\tilde\theta_L - \chi(g)/[N-2]\big)\ , \quad -\tilde B_L < \tilde\theta_R < \infty\ ,\\
\rho_0(\zeta) &= \lim_{N\to 2} \frac{1}{N-2}\, \rho\big(\zeta/[N-2]\big)\ , \quad -\chi(g) < \zeta < \chi(g)\ .
\end{aligned} \tag{48}$$

Because the zero modes occupy a range of θ of order $(N-2)^{-1}$, their density must be of order $N-2$, and so we have included a compensating factor in the definition of ρ_0.

The right- and left-moving Bethe ansatz equations are exactly as in Eq. (8), using K_{I} constructed from S_{I}. In particular, the zero-modes do not enter into these equations because the rapidity difference is large and $\partial_\theta S_{\mathrm{II}}$ is of order $N-2$. To write the zero mode equations we define

$$\begin{aligned}
\frac{1}{2\pi i}\, \partial_\zeta \ln S_{\mathrm{II}}(\zeta) = K_{\mathrm{II}}(\zeta) &= \frac{1}{2}\,\delta(\zeta) + k(\zeta)\ ,\\
k(\zeta) &= \frac{1}{4\pi^2 + \zeta^2}\ .
\end{aligned} \tag{49}$$

[c] We can divide the rapidity range so that right-movers have $\theta > \chi - \epsilon^{-1}$, left-movers have $\theta < -\chi + \epsilon^{-1}$, and zero modes are in between. As long as ϵ goes to zero as $N \to 2$ but does so more slowly than $N-2$ itself (e.g. $\epsilon = \sqrt{N-2}$), one gets the indicated ranges for $\tilde\theta_R$, $\tilde\theta_L$, and ζ.

Then

$$\int_{-\chi}^{\chi} k(\zeta-\zeta')\rho_0(\zeta')d\zeta' + 2\pi\big[k(\zeta-\chi)\mathcal{Q}_R + k(\zeta+\chi)\mathcal{Q}_L\big] = \frac{1}{2}\,\rho_0(\zeta)\ ,$$
$$-\chi<\zeta<\chi\ . \qquad (50)$$

We use $\mathcal{Q}_{R,L}$ here to distinguish these from the densities $\mathcal{J}_{R,L}$ of the fermionic description. The coupling g now enters into the Bethe ansatz equations only through the implicit g-dependence of the rapidity range χ.

The Bethe ansatz equations for ρ_R and for ρ_L separate from the other components of ρ, while the total $\mathcal{Q}_R$ and $\mathcal{Q}_L$ give rise to inhomogeneous terms in the ρ_0 equation. The zero modes do feed back into the undifferentiated Bethe equation (33) which determines the total $\mathcal{Q}_R$ and $\mathcal{Q}_L$. The energy of the zero modes is exponentially small in the limit, so the energy and momentum come only from the right- and left-movers as in Eqs. (19, 20):

$$\mathcal{E}+\mathcal{P} = \pi\mathcal{Q}_R^2\ , \qquad \mathcal{E}-\mathcal{P} = \pi\mathcal{Q}_L^2\ . \qquad (51)$$

The zero modes affect the energy indirectly because they enter into the determination of $\mathcal{Q}_R$ and $\mathcal{Q}_L$.

As in the previous section, the Bethe ansatz has been derived by taking the limit of a state with particles only, but it can be extended to negative rapidity densities. Using the expressions in Ref. [39], the particle-antiparticle reflection amplitude $S_R = \sigma_1^+ + \sigma_3^+$ vanishes for rapidities of order $1/(N-2)$, while the particle-antiparticle transmission amplitude $S_T = \sigma_1^+ + \sigma_2^+$ gives a kernel which is equal to $-k(\zeta)$. Therefore we can use the Bethe ansatz equations (8, 50) freely for positive and negative densities as long as particle and antiparticles are separated by rapidities of order $1/(N-2)$. That is, all the right-movers must be of one type, and all the left-movers similarly, but the zero mode density may have a sign that changes as a function of rapidity, since the typical rapidity difference for the zero modes is of order $1/(N-2)$.

The zero modes represent a substantial complication, because their rapidity support is bounded in both directions. In fact, the kernel k is essentially the same as that for the nonconformal $O(3)$ model, and so the zero mode equation can only be solved as a series in χ or in $1/\chi$ [19]. We see from the perturbative calculation (42) that the expansion in $1/\chi$ corresponds to small g in the nonlinear sigma model. In the other limit, $\chi\to 0$, the zero modes disappear. Taking the case that $\mathcal{Q}_R = \mathcal{Q}_L \equiv \mathcal{Q}/2$, we have $\mathcal{E} = \pi\mathcal{Q}^2/4$. On the other hand, since the zero modes carry no charge in this limit we can identify $\mathcal{Q}$ with the Noether charge Π, in terms of which $\mathcal{E} = g^2\Pi^2/2$. Thus we identify $g^2 = \pi/2$ with $\chi \to 0$. The $N \to 2$ limit therefore can reach the range of couplings

$$0 < g^2 \leq \frac{\pi}{2}\ , \qquad \infty > \chi \geq 0\ . \tag{52}$$

It is curious that we cannot reach all values of the coupling g; perhaps there is some extension or continuation of our construction that makes it possible to do so.

The expansion for small χ is straightforward because the integral term in the zero mode equation (50) is small in this limit and the equation can be solved iteratively; also the kernel k can be expanded in ζ, which is of order χ. One readily obtains

$$\mathcal{Q}_0 = (\mathcal{Q}_R + \mathcal{Q}_L)\left\{\frac{\chi}{\pi^2} + \frac{\chi^2}{\pi^4} + \left[1 - \frac{\pi^2}{3}\right]\frac{\chi^3}{\pi^6} + \left[1 - \frac{\pi^2}{2}\right]\frac{\chi^4}{\pi^8} + \ldots\right\}\ . \tag{53}$$

Identifying the total Noether charge $\mathcal{Q}_0 + \mathcal{Q}_R + \mathcal{Q}_L = \Pi$, we can also write

$$\mathcal{Q}_R + \mathcal{Q}_L = \Pi\left\{1 - \frac{\chi}{\pi^2} + \frac{\chi^3}{3\pi^4} - \frac{\chi^4}{6\pi^6} + \ldots\right\}\ . \tag{54}$$

If we take again the state with $\mathcal{Q}_R = \mathcal{Q}_L = \mathcal{Q}/2$, matching the energies $\pi\mathcal{Q}^2/4 = g^2\Pi^2/2$ as in the previous paragraph gives

$$g^2 = \frac{\pi}{2}\left\{1 - 2\,\frac{\chi}{\pi^2} + \frac{\chi^2}{\pi^4} + \frac{2\chi^3}{3\pi^4} - \frac{\chi^4}{\pi^6} + \ldots\right\}\ . \tag{55}$$

Thus we obtain the functional relationship between the coupling (radius) in the $O(2)$ theory and the parameter χ which governs the $N \to 2$ limit of the rapidity difference between the right- and left-movers, in the neighborhood of $g^2 \sim \pi/2$.

For states with $\mathcal{Q}_R \neq \mathcal{Q}_L$ we could again use the undifferentiated equation (33) in order to identify the quantum numbers, but a simple shortcut is to notice that the momentum density $\mathcal{P} = \pi(\mathcal{Q}_R^2 - \mathcal{Q}_L^2)/2 = \Pi\phi'$ is adiabatically invariant. We then have

$$\begin{aligned}\mathcal{Q}_R - \mathcal{Q}_L &= \frac{2\phi'}{\pi}\,\frac{\Pi}{\mathcal{Q}_R + \mathcal{Q}_L} = \frac{2\phi'}{\pi}\left[1 + \frac{\mathcal{Q}_0}{\mathcal{Q}_R + \mathcal{Q}_L}\right] \\ &= \frac{2\phi'}{\pi}\left\{1 + \frac{\chi}{\pi^2} + \frac{\chi^2}{\pi^4} + \left[1 - \frac{\pi^2}{3}\right]\frac{\chi^3}{\pi^6} + \left[1 - \frac{\pi^2}{2}\right]\frac{\chi^4}{\pi^8} + \ldots\right\}\ . \end{aligned}\tag{56}$$

The $1/\chi$ expansion is more involved, and the conformal limit appears no simpler than the general case. We therefore simply take the $N \to 2$ limit of the known $O(N)$ result. Using Eqs. (13, 22, 23) of Ref. [20], with $\Pi = \partial f/\partial h$ and $\chi = (N-2)B$, gives

$$\frac{\mathcal{E}}{\Pi^2} = \frac{\pi}{\chi} - \frac{\pi}{\chi^2}\ln 8\chi + \dots \ . \tag{57}$$

Equating this to the canonical result $g^2/2$ and solving for χ leads to

$$\chi = \frac{2\pi}{g^2} + \ln g^2 + \ln\frac{4}{\pi} + \dots \ . \tag{58}$$

In particular, this reproduces the two-loop result (42).

The results (55, 58) just give the relation between the parameter g of the canonical description and the parameter χ of the integrable description. Once this relation is known, one can use the integrable description to calculate physical quantities, such as the spectrum of excitations. Of course, in this case the canonical description is vastly simpler.

5. The $OSp(2+2M|2M)$ Coset Model

In Sec. 3 we solved free field theory in a difficult way, and then in Sec. 4 we solved it in an even more difficult way. We can now take these efforts and apply them rather directly to a conformal theory which is not free, and not solvable by the usual methods of chiral algebra. The $OSp(N+2M|2M)$ coset model has the action

$$S = -\frac{1}{2g^2}\int d^2x\, J_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j \ , \quad J_{ij}\varphi^i\varphi^j = 1 \ . \tag{59}$$

The components φ^i have statistics

$$\begin{aligned} \text{commuting}: &\quad 1 \le i \le N+2M \ , \\ \text{anticommuting}: &\quad N+2M+1 \le i \le N+4M \ , \end{aligned} \tag{60}$$

and J_{ij} is

$$J = \begin{bmatrix} I_{N+2M} & 0 & 0 \\ 0 & 0 & -I_M \\ 0 & I_M & 0 \end{bmatrix} \ . \tag{61}$$

Consider an amplitude in which only the first N bosonic fields are present in the external states and operators. The remaining $2M$ bosonic fields and the $2M$ fermionic fields appear only in loops, and by drawing the graphs in single-line notation (or introducing an auxiliary field to make the integral over φ gaussian) it becomes evident that these M-dependent contributions cancel. Thus these amplitudes are independent of M, and are the same in the supergroup model as in the bosonic $O(N)$ sigma model [40,41]. In particular, the S-matrix for states involving only the first N bosonic components is identical to the $O(N)$ result (43). But then there is a unique $OSp(N+2M|2M)$-invariant extension [16],

$$|k\,\theta, l\,\theta'; \mathrm{in}\rangle = S_{kl,ij}(\theta-\theta')|i\,\theta, j\,\theta'; \mathrm{out}\rangle$$
$$S_{kl,ij}(\theta) = J_{ij}J^{-1}_{kl}\sigma^+_1(\theta) + \delta_{ik}\delta_{jl}\sigma^+_2(\theta) + (-1)^{p_i+p_j}\delta_{il}\delta_{jk}\sigma^+_3(\theta)\ , \quad (62)$$

where p_i is 0 when φ^i is bosonic and 1 when it is fermionic. This S-matrix is not unitary, but preserves an indefinite inner product built from J.

We can now take the $N \to 2$ limit as before, and in this way obtain the Bethe ansatz for the conformally invariant $OSp(2{+}2M|2M)$ coset. The finite field calculations in Ref. [19, 20] and in Sec. 4 involve only one $O(2)$ charge and so lift directly to the $OSp(2{+}2M|2M)$ coset [16] — for states with only a single $O(2) \subset O(2{+}2M)$ charge the energy reduces to that of the $O(2)$ theory and so can be calculated in free field theory. Now, however, we can go on to consider more general states, having charges in more than one $O(2)$ subgroup of $OSp(2{+}2M|2M)$. We will leave the detailed study of these states for future work.

Why not can we not lift the simpler massless Thirring S-matrix of Sec. 3 to $OSp(2{+}2M|2M)$? The difficulty is that the Thirring fermions have no simple transformation property under $OSp(2{+}2M|2M)$. They are spinors under the first $O(2) \subset O(2{+}2M)$ but are neutral under the commuting $O(2)$'s, so they do not even lift to a spinor representation of $OSp(2{+}2M|2M)$. Thus the Thirring description does not seem useful for the supercoset.[d]

6. Discussion

We have found (Sec. 2, 3) that for some integrable theories, the conformal limit leads to simplifications in the Bethe ansatz. For others (Sec. 4) it does not. Unfortunately, the supergroup coset $OSp(2{+}2M|2M)$ appears to be of the latter type. It is conceivable that there is a simpler description of this model; different massive perturbations of a given conformal theory define different bases of states and so different S-matrices. However, we suspect that in the present case there may simply be a certain irreducible

[d] R. Roiban has independently pointed out another difficulty. If we simply lift the Thirring S-matrix as in Eq. (62), treating the Thirring fermions as vectors of $OSp(2{+}2M|2M)$, it does not satisfy the Yang-Baxter equation. The existence of an S-matrix (23) containing a free parameter γ depends on identities that are special to $O(2)$ and do not lift to $OSp(2{+}2M|2M)$.

complexity to the integral equations that must be solved.

Since this work is ultimately directed at a better understanding of the AdS/CFT duality, let us list the steps that would be needed to reach this goal. First, one must find the S-matrix having the appropriate symmetry, for example $PSU(2,2|4)$, and the appropriate degrees of freedom. This S-matrix is likely to be similar in complexity to the $OSp(2{+}2M|2M)$ S-matrix. In our case we were aided by having a family of nonconformal theories whose limit we could take; perhaps given this example one can determine the S-matrix for other conformal supergroup theories directly. Second, one must understand how the BRST ghost degrees of freedom of the world-sheet theory enter into the integrable description, and how the BRST charge acts on the states in the integrable description.

Third, to obtain a complete integrable description of the spectrum one must go beyond the large-field case and understand finite volume effects. The problem is that continuum S-matrices such as those that we have used are defined with reference to the infinite-volume vacuum. Putting the system in a finite volume changes the vacuum (for example there is a Casimir energy) and so changes the excitation energies and the S-matrix.[e] This poses a great difficulty, and so for the most part finite volume energies are understood only for the ground state [42] and for twisted sector ground states [43–46], via a world-sheet space-time duality and the Thermodynamic Bethe Ansatz (see however Ref. [47]). The other approach to finite volume energies is to find a discretized version of the integrable theory, for which there is a trivial ferromagnetic vacuum, and build the theory around this vacuum (for work on discrete supercoset models see ref. [15]). We had hoped that the large-charge states such as those that we are considering could play the role of such a ferromagnetic vacuum but directly in the continuum theory. However, these still have low-lying excitations and long-range correlations, and so they are not as simple as would be needed.

For the $OSp(2+2M|2M)$, or its continuation to $OSp(2,2M|2M)$ there is the interesting question as to whether the large-curvature limit $g^2 \to \infty$ has any simple dual description; in the AdS/CFT case this is the limit where the dual field theory becomes weakly coupled. Finally, we believe that our most interesting result is the nature of the $N \to 2$ limit of the

[e] One gets wrong answers if one simply ignores this and forges ahead. The simplest example is the state of a single fermion with n units of right-moving momentum. The naive Bethe ansatz would give energy $2\pi n/L$ independent of g (this is trivial, since the S-matrix does not enter). On the other hand, the CFT calculation using has explicit g-dependent terms involving the fermion charges (or the momentum and winding, in bosonized form).

$O(N)$ and $OSp(N+2M|2M)$ theories: this limit seems to be sensible, but has a nontrivial zero mode sector in addition to the right- and left-movers. Some of the features that we have found may arise in other approaches to the integrability of supergroup models.

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