CONFORMAL FIELD THEORY AT CENTRAL CHARGE $c = 0$ AND TWO-DIMENSIONAL CRITICAL SYSTEMS WITH QUENCHED DISORDER

V. GURARIE

Department of Physics, CB390
University of Colorado
Boulder CO 80309

A. W. W. LUDWIG

Department of Physics
University of California
Santa Barbara CA 93106

We examine two-dimensional conformal field theories (CFTs) at central charge $c = 0$. These arise typically in the description of critical systems with quenched disorder, but also in other contexts including dilute self-avoiding polymers and percolation. We show that such CFTs must in general possess, in addition to their stress energy tensor $T(z)$, an extra field whose holomorphic part, $t(z)$, has conformal weight two. The singular part of the Operator Product Expansion (OPE) between $T(z)$ and $t(z)$ is uniquely fixed up to a single number b, defining a new 'anomaly' which is a characteristic of any $c = 0$ CFT, and which may be used to distinguish between different such CFTs. The extra field $t(z)$ is not primary (unless $b = 0$), and is a so-called 'logarithmic operator' except in special cases which include affine (Kač–Moody) Lie-super current algebras. The number b controls the question of whether Virasoro null-vectors arising at certain conformal weights contained in the $c = 0$ Kač table may be set to zero or not, in these nonunitary theories. This has, in the familiar manner, implications on the existence of differential equations satisfied by conformal blocks involving primary operators with Kač-table dimensions. It is shown that $c = 0$ theories where $t(z)$ is logarithmic, contain, besides T and t, additional fields with conformal weight two. If the latter are a fermionic pair, the OPEs between the holomorphic parts of all these conformal weight-two operators are automatically covariant under a global $U(1|1)$ supersymmetry. A full extension of the Virasoro algebra by the Laurent modes of these extra conformal weight-two fields, including $t(z)$, remains an interesting question for future work.

Table of Contents

1. Introduction

In the last four decades remarkable progress has been made in our understanding of second order phase transitions. Beginning with the scaling hypothesis put forward in the sixties and continuing with the subsequent development of the renormalization group methods, many questions regarding the properties of matter in the vicinity of critical points have been thoroughly answered. A further milestone was set in 1984 with the development of conformal field theory (CFT) by A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov [1, 2]. Indeed, the methods of CFT provided access, in a completely nonperturbative manner, to a vast variety of problems involving second order phase transitions of classical statistical mechanics in two dimensions. The various exact solutions can be classified, and in part distinguished, by a parameter called central charge c. Physically the central charge measures the response of a scale invariant (or critical) field theory in two dimensions to a change of the geometry of the space on which it lives. Equivalently, the central charge is universally related to the coefficient of the length dependence of the ground state (or Casimir) energy of a critical $(1+1)$-dimensional relativistic field theory living on a space of finite length L with periodic boundary conditions, i.e. where space-time is a cylinder. (These results are due to W. J. Blöte, J. L. Cardy and M. P. Nightingale [3] and simultaneously I. Affleck [4].)

Once the central charge c of a given critical physical system is known, the techniques of CFT make it possible, in many important cases, to calculate exactly all its correlation functions in a quite straightforward way [1, 2]. At the same time, finding the central charge of a given system may not be entirely obvious. In some cases the central charge can be obtained by elementary computation. This is the case for example for the Ising model, whose central charge $c = 1/2$ can be obtained using the free Majorana fermion representation. In other cases, the central charge of a system can be found from symmetry analyses, as is the case e.g. for Wess-Zumino-Witten (WZW) models [2, 5], which have found many applications in Condensed Matter Physics, including e.g. one-dimensional Quantum Spin Chains [6], the Kondo effect [7], topological Quantum Computation [8], and many others. If not known analytically, the central charge of a given system can also be determined numerically, using the finite size scaling methods mentioned above.

However, there exists a class of problems where knowing the central charge tells us close to nothing about the solution of the CFT. These are problems where second order phase transitions happen (in two dimensions) in the presence of quenched disorder. Critical field theories describing such problems can be shown to typically have vanishing central charge, $c = 0$. It turns out, unlike in the case of their pure (i.e. disorderless) counterparts which have $c \neq 0$, that the knowledge of their central charge $c = 0$ does not contribute much to the solution of these theories. Indeed, there is a large variety of CFTs with vanishing central charge, each of which corresponds to a different critical point. And with very few exceptions little is known about these theories.

These theories arise for example in the description of disordered electronic systems.[a] Consider a quantum mechanical particle moving in a random potential in d dimensions. The system is described by a Hamiltonian

$$H = H_0 + V(x), \quad H_0 = -\frac{\hbar^2}{2m}\nabla^2, \tag{1.1}$$

where x denotes d-dimensional space, and $V(x)$ is a random, time-independent potential. When describing universal critical properties, the latter may often be taken, without loss of generality, to have a probability distribution which is a short-ranged Gaussian with zero mean

$$\langle V \rangle = 0\,, \qquad \langle V(x)V(y) \rangle = \lambda\ \delta(x-y)\,. \tag{1.2}$$

where $\langle ... \rangle$ denotes the average over all configurations of the disorder potential $V(x)$.

All relevant information concerning the motion of a quantum particle can be extracted from its (advanced or retarded) Green's functions [b]

$$G^{\pm}(E)(x,y) = \left(x \left| \frac{1}{E - H_0 - V \pm i\epsilon} \right| y \right), \qquad (\epsilon > 0)\,. \tag{1.3}$$

These can be calculated with the help of the following Gaussian functional integral, involving a complex [c] scalar field $\phi(x)$, $\bar{\phi}(x)$

$$\begin{aligned}(\pm i)G^{\pm}(E)(x,y) = \frac{1}{Z}\int D\bar{\phi}D\phi\ \phi(x)\bar{\phi}(y)\\ \times \exp\left[\pm i \int d^d x\ \bar{\phi}\,(E - H_0 - V \pm i\epsilon)\,\phi\right]\,,\end{aligned} \tag{1.4}$$

[a] In the absence of electron-electron interactions, or when these are irrelevant in the renormalization group sense.

[b] We use round brackets for Dirac's bra and ket symbols, to distinguish them from the averaging symbols "$\langle$" and "$\rangle$".

[c] We use the notation $\bar{\phi}(x) := \phi^*(x)$ for the complex scalar field.

where d is the dimensionality of space, and Z is the partition function

$$Z = \int D\bar{\phi} D\phi \quad \exp\left[\pm i \int d^d x \ \bar{\phi}\,(E - H_0 - V \pm i\epsilon)\,\phi\right].$$

The plus or minus sign in the exponential is chosen to insure convergence. Although this maps the problem of a Green's function of a quantum particle moving in a random potential into a correlation function of a field theory, this field theory is intractable, as written. Indeed, the correlation function has to be computed for any arbitrary random function $V(x)$, and the corresponding field theory is not even translationally invariant. If, however, we concentrate on computing Green's functions which are averaged over all disorder configurations $V(x)$, further progress is possible. It is not practical, of course, to average (1.4) over random $V(x)$ directly, because of the factor $1/Z$ in (1.4), where the 'partition function' Z is itself a random variable. Instead, one can employ one of the following two 'tricks', commonly referred to as '*replica*'- and '*supersymmetry tricks*'. In the present paper we concentrate mostly on the supersymmetry trick, which involves rewriting the denominator of (1.4) as a functional integral over anticommuting (Grassmann) variables ψ, $\bar{\psi}$

$$\frac{1}{Z} = \int D\bar{\psi} D\psi \quad \exp\left[\pm i \int d^d x \ \bar{\psi}\,(E - H_0 - V \pm i\epsilon)\,\psi\right]. \qquad (1.5)$$

This brings the Green's function into the form

$$(\pm i)\ G^{\pm}(E)(x,y) = \int D\bar{\phi} D\phi D\bar{\psi} D\psi \ \phi(x)\bar{\phi}(y)\ e^{-S_V}, \qquad (1.6)$$

where the action S_V is given by

$$S_V = \mp i \int d^d x \ \left\{\bar{\phi}\,(E - H_0 - V \pm i\epsilon)\,\phi + \bar{\psi}(E - H_0 - V \pm i\epsilon)\psi\right\}. \qquad (1.7)$$

The total partition function is unity, because the fermionic and bosonic contributions to it cancel. Now averaging over the random potential becomes possible with the help of the standard Gaussian identity

$$\left\langle \exp\left[i \int d^d x \ V(x) J(x)\right] \right\rangle = \exp\left[-\frac{\lambda}{2} \int d^d x \ J^2(x)\right]$$

valid for an arbitrary function $J(x)$. This yields

$$(\pm i)\ \langle\, G^{\pm}(E)(x,y)\,\rangle = \int D\bar{\phi} D\phi D\bar{\psi} D\psi \ \phi(x)\bar{\phi}(y)\ e^{-S}, \qquad (1.8)$$

where the action S is given by

$$S = \mp i\int d^d x \left\{ \bar{\phi}\,(E - H_0 \pm i\epsilon)\,\phi + \bar{\psi}(E - H_0 \pm i\epsilon)\psi \pm i\,\frac{\lambda}{2}\left(\bar{\phi}\phi + \bar{\psi}\psi\right)^2 \right\}, \qquad (\lambda > 0)\,. \tag{1.9}$$

In summary, the problem of computing an averaged Green's function of a quantum mechanical particle in a random potential in d dimensional space, has been mapped into a problem of computing a correlation function in a d-dimensional field theory of interacting bosonic and fermionic degrees of freedom. As we already mentioned, (1.8) is often referred to as the supersymmetry (SUSY) approach to disordered systems.[d]

Theories of this kind have been extensively studied in the literature [9], using a variety of techniques in various dimensionalities. Progress in accessing critical properties can sometimes be made if a small parameter is available, such as for example in the $d = 2 + \epsilon$ expansion. The topic of this paper is two-dimensional physics, and here a small parameter is typically unavailable. Accordingly, one needs to rely on nonperturbative techniques, and CFT is expected to provide such tools. To be specific, let us discuss in a little more detail a specific disordered two-dimensional electronic system known to possess a critical point. Consider a quantum mechanical particle moving in a plane (coordinate x) in the presence of a perpendicular constant magnetic field and in a random potential $V(x)$. In order to write down an effective field theory for this problem, we proceed as above, but now choosing H_0 to be the Hamiltonian for a free particle in $d = 2$ dimensions, moving in a constant magnetic field

$$H_0 = -\frac{\hbar^2}{2m}\sum_j \left(\frac{\partial}{\partial x_j} + \sum_k \frac{i\epsilon_{jk}x_k}{2l^2} \right)^2 ,$$

where l is the magnetic length.

It also turns out that the disorder averaged Green's function $\langle\, G^{\pm}(E)\,\rangle$, as in Eq. (1.8), does not exhibit any critical behavior whatsoever, in this, and in the other problems discussed above. It decays exponentially on distances larger than the particle's mean free path. However, the average of

[d] It should be emphasized that the action S does not possess space-time SUSY (where the translation operator is a suitable square of the supercharge), the SUSY that is usually understood in high energy physics. Rather, it has an isotopic SUSY which involves rotating bosonic and fermionic fields ϕ and ψ into each other (see e.g. [9]), and is often referred to as supergroup symmetry.

the advanced/retarded product

$$\left\langle G^{+}(E)\, G^{-}(E) \right\rangle \tag{1.10}$$

can be critical [10]. Such a product can also be cast into the form of a correlation function in a field theory if only one chooses two independent functional integral representations such as (1.6) for the two Green's functions involved in the product, subject to the same disorder potential, and then averages over disorder $V(x)$. The resulting field theory is similar to (1.8) but contains two copies of each of the two basic fields ϕ (bosonic) and ψ (fermionic). As the parameter E (energy) is adjusted, the resulting field theory goes through a critical point, called the Integer Quantum Hall plateau transition. This transition is experimentally observed [e] in the Integer Quantum Hall Effect (see e.g. [12, 13]). Even though much is known from numerical work [14] about the critical properties [f] of this transition (in the absence of interactions), and even though a theoretical description in terms of a nonlinear sigma model with topological term [15, 16] was given a long time ago, an analytical solution of the transition has been lacking for, by now, about two decades. As already mentioned, this is due to the fact that this problem lacks a small parameter, and a genuinely nonperturbative approach is unavoidable; conformal field theory is expected to provide such a nonperturbative tool. Nevertheless, CFT techniques have not yielded a solution to this problem, to date. This is due to certain 'technical' difficulties which CFTs, aimed at describing disordered critical points, present. It is because of these 'difficulties' that exact, nonperturbative solutions of the infinite number of constraints imposed by the conformal symmetry group, have not been forthcoming as readily as was the case in pure (i.e. nonrandom) critical theories [1]. Some of these difficulties are the subject of this paper.

Even though an analytical solution of the Integer Quantum Hall plateau transition is still lacking as of today, it has been possible, fairly recently, to find an analytical solution of the rather similar (but not identical) problem of the so-called Spin Quantum Hall Effect (SQHE) plateau transition [17]. The resulting theory is a supersymmetric formulation of the 2D percolation problem.[g] Percolation and the problem of dilute self-avoiding walks, in fact,

[e] Even though long-range Coulomb interactions between the electrons appear to modify the transition, unless they are screened by hand, in which case they are known to leave the non-interacting universality class unaffected (see e.g. Ref.'s [11, 12]).

[f] For example, the correlation length exponent is known to be numerically close to $\nu = 7/3$.

[g] Another solution of the SQHE transition, not based on SUSY, was later found in Ref. [18].

are two of the best understood disordered systems in two dimensions. In spite of this, the nature of their CFT, including for example multi-point correlation functions, is quite poorly understood [19]. Both systems have central charge $c = 0$, and we will describe aspects of their CFT below. Because the self-avoiding polymer problem is (in a formal sense) closely related to our formulation of particle localization in (1.9), let us describe this now in some detail. The statistics of self-avoiding dilute polymer chains in d dimensions can be described by the following SUSY Landau–Ginzburg action [h] due to Parisi and Sourlas [20]

$$S = \int d^d x \ \left\{ \bar{\phi} \left(H_0 - E \right) \phi + \bar{\psi} \left(H_0 - E \right) \psi + \frac{g}{2} \left(\bar{\phi}\phi + \bar{\psi}\psi \right)^2 \right\}, \quad (g > 0) \tag{1.11}$$

with H_0 as in (1.1) with $\hbar = m = 1$. Note that, in contrast to the problem of the motion of a quantum particle, described by the theory (1.9), the 'convergence factors' $\pm i\epsilon$ have disappeared, and it turns out that the analog of the 'single-particle Green's function',

$$\left\langle G^{\pm}(E)(x,y) \right\rangle = \int D\bar{\phi} D\phi D\bar{\psi} D\psi \ \phi(x)\bar{\phi}(y) \ e^{-S}, \tag{1.12}$$

now exhibits critical behavior (has power-law decay, at $E = 0$) and characterizes the statistics of a polymer chain with end points fixed at positions x and y.

As was already mentioned, conformal field theories describing disordered critical points in 2D typically have central charge $c = 0$. Indeed, as emphasized below (1.7), these theories are constructed in such a way that their partition function is always exactly equal to unity,

$$\int D\bar{\phi} D\phi D\bar{\psi} D\psi \ \ e^{-S} = 1,$$

as a consequence of exact cancellation of bosonic and fermionic integrals. The free energy is therefore exactly zero. This is also true when the theory is defined on a cylinder. Hence the central charge vanishes.

Let us end our introductory remarks by briefly mentioning the so-called replica approach to disordered systems [21]. This involves introducing, before taking the average over disorder realizations, several copies of, say, the commuting field ϕ_α, $(\alpha = 1, ..., n)$, instead of introducing the anticommuting

[h] Formally, one might envision this action as arising from (1.9) by analytic continuation, and a change of sign of λ, i.e. $g := (-\lambda) > 0$, but we will not pursue this here.

field ψ, and then taking the number n of copies to zero ($n \to 0$) ('Bosonic Replicas'). (An equivalent formulation can be obtained by using n copies of the anticommuting (Grassmann) field, and no commuting fields ('Fermionic Replicas').) For example, introducing Bosonic replicas in (1.4), which describes the Green's function of a particle moving in a random potential, and performing the average over disorder, one easily finds that the following functional integral can be used as an alternative to (1.8),

$$(\pm i) \left\langle G^{\pm}(E)(x,y) \right\rangle = \lim_{n\to 0} \int \prod_{\alpha=1}^{n} \left[D\bar{\phi}_\alpha D\phi_\alpha \right] \phi_1(x)\bar{\phi}_1(y)\ e^{-S_r(n)}, \quad (1.13)$$

where the 'replicated action' $S_r(n)$ is given by

$$S_r(n) = \mp i \int d^d x \left[\sum_{\alpha=1}^{n} \bar{\phi}_\alpha \left(E - H_0 \pm i\epsilon\right) \phi_\alpha \pm i \frac{\lambda}{2} \left(\sum_{\alpha=1}^{n} \bar{\phi}_\alpha \phi_\alpha \right)^2 \right]. \quad (1.14)$$

Calculating the Green's function (1.13) now involves doing the functional integral at arbitrary integer n and then analytically continuing the answer to $n \to 0$. In fact, the same comments as those given after (1.10) in the SUSY context apply here, and a duplication of the so-far introduced variables is required for the quantum particle in a random potential, but we refrain here from writing out the details. (The low energy effective theories are in fact nonlinear sigma models [10], both in the replica and the SUSY descriptions.)

The replica method is easily used in perturbative calculations, where the number n of copies typically appears in the form of a polynomial in n, in any order in perturbation theory. This is easily, and unambiguously, continued to $n \to 0$. In the context of a nonperturbative analysis, one would, at least naively, need a critical theory for all (large) integer values of n, each of which would have a central charge $c(n)$. This may (but typically will not uniquely) determine an analytic continuation into $n \to 0$. Such an approach is known not to be feasible for the 2D theory describing the Integer Quantum Hall plateau transition discussed above. On the other hand, the dilute self-avoiding polymer problem is known to be described (in any dimensionality d), due to P. G. deGennes [22], as the $n \to 0$ limit of the replica analog [i] of the SUSY action (1.11). In $d = 2$ dimensions, a number of properties of this replica action can be obtained exactly [23, 24] in the continuous range $-2 \leq n \leq +2$ of the parameter n, and this model is often referred to as the $O(n)$ model. A similar analysis and corresponding results exist also for the

[i] Which bears the same relationship to (1.14), that (1.11) has with (1.9).

2D q-state Potts model in the continuous parameter range $0 \leq q \leq 4$. The $q \to 1$ limit of the q-state Potts is known [25] to describe percolation.

Although the SUSY technique is better controlled than the replica approach, it is limited to non-interacting random systems. This is because the SUSY technique is based crucially on the ability to represent the inverse partition function, such as (1.5), in terms of a fermionic functional integral. This is only possible if the original problem without disorder did not contain interactions (non-Gaussian terms). A much-studied example of a disordered classical 2D statistical mechanics system which *is interacting* is provided by the random-bond q-state Potts model. It can be analyzed [26] with the help of the replica trick in an expansion in $(q-2)$ about the Ising case $(q = 2)$.

This paper contains attempts by the authors to understand in more detail conformal field theories at central charge $c = 0$. Our prime motivation arises from the desire to understand better the structure of CFT underlying two-dimensional disordered critical points. There is a significant number of such critical points which are of great physical interest but which are typically poorly understood. (Some have been mentioned above.) In particular, we give here a pedagogical and detailed exposition of results which appeared earlier in Ref. [27], but we also present a variety of new, so-far unpublished results.

Specifically, we review certain unusual features, which distinguish $c = 0$ conformal theories from ordinary, say unitary CFTs. One of the most dramatic such features is the indecomposability, or 'logarithmic' structure which typically (except in certain special cases, including affine current algebras) appears in the identity representation of the Virasoro algebra at $c = 0$. This manifests itself through the appearance of a so-called 'logarithmic partner' $t(z)$ of the stress energy tensor. Moreover, $c = 0$ CFTs possess a novel 'anomaly' number sometimes denoted by b, which plays, in some sense, a role similar to the central charge in $c \neq 0$ theories: the parameter b may be used to distinguish different $c = 0$ theories. These general properties of a $c = 0$ CFT are discussed in Section 2, where we also motivate and derive the fundamental OPE between the (ordinary) stress tensor and its logarithmic partner.

An important role is often played in CFT by so-called null-vectors (or: singular vectors). These are Virasoro descendants which are themselves primary. They are known to occur when primary operators have conformal weights contained in the Kač table (here at $c = 0$). It is important to know if such a null-vector can be set to zero, because in that case correlation functions involving the Kač-table operator will satisfy differential equations, which makes them easily computable. While in an ordinary (unitary) CFT null-vectors are always set to zero, this is not necessarily the case in a nonunitary theory, like a $c = 0$ CFT.

In Section 3 we make a connection between the 'anomaly' number b and Kač null-vectors. Interestingly, the number b controls the question as to whether certain Kač-table null-vectors vanish identically or not, and hence whether certain correlation functions satisfy the corresponding differential equations. We discuss the cases of Kač-table operators with nonvanishing two-point functions, first those with nonvanishing, and subsequently those with vanishing conformal weights.

In Section 4 we review aspects of critical disordered systems described by the supersymmetry method. In these theories, a partner of the stress tensor, of the kind that appeared in Section 2 entirely from considerations of conformal symmetry, emerges naturally on grounds of supersymmetry. Moreover, a pair of conformal weight-two fermionic operators appears together with the stress tensor and its partner in the same supersymmetry multiplet.

In Section 5 we show that based purely on conformal symmetry considerations, there must exist additional fields, besides the stress tensor $T(z)$ and its partner $t(z)$, whose holomorphic parts have conformal weight two, if $t(z)$ is 'logarithmic'. Under the only assumption that these additional fields form a fermionic (anticommuting) pair, we show that the (holomorphic) OPEs between all these weight-two fields are automatically covariant under a global $U(1|1)$ SUSY.

We end the main part of the paper by comments and speculations about a possible extended chiral symmetry in $c = 0$ CFT, based on the notions developed here.

Three appendices provide a number of technical details. We show in Appendix A that the anomaly number b must be unique in a given theory (as discussed at the beginning of Subsection 3.1). Appendix B addresses details of the computation of the OPE of descendant operators in the $c = 0$ CFT possessing the logarithmic features discussed in this paper. We also demonstrate in Section B.9 the complete subtraction of logarithms to all orders in the OPE (5.4). Appendix C addresses certain details referring to the footnote below (3.12) in Subsection 3.1.

2. Conformal Field Theory at $c = 0$

2.1. $c \to 0$ *catastrophe*

Conformal field theories (CFTs) with central charge $c = 0$ are very different from those with $c \neq 0$. Consider a CFT with central charge c and a primary scalar operator $A(z, \bar{z})$ with left/right conformal weights $(h, \bar{h})$, $h = \bar{h}$, and nonvanishing[j] two-point function. We choose to consider operators whose two-point functions are normalized to unity,[k]

$$\left\langle A(z, \bar{z}) A^{\dagger}(0, 0) \right\rangle = \frac{1}{z^{2h} \bar{z}^{2h}} . \tag{2.1}$$

The operator product expansion (OPE) of this operator with its conjugate is known [28] to be given by

$$\begin{aligned} A(z, \bar{z}) A^{\dagger}(0, 0) = \frac{1}{z^{2h}} \left(1 + \frac{2h}{c} z^2 T(0) + \ldots \right) \frac{1}{\bar{z}^{2h}} \left(1 + \frac{2h}{c} \bar{z}^2 \bar{T}(0) + \ldots \right) \\ + \text{other primaries} , \end{aligned} \tag{2.2}$$

where $T(z)$, and $\bar{T}(\bar{z})$ are the holomorphic and antiholomorphic components of the stress-energy tensor of the theory. Here, '*other primaries*' denotes possible contributions to the OPE from primary operators other than the identity operator. From now on, in the rest of this paper, we will focus entirely, as is customary, on the holomorphic dependence, with the understanding that a suitable 'gluing' with the antiholomorphic dependence has to be performed at the end to obtain bulk correlation functions. With this understanding, we write the OPE (2.2) as

$$A(z) A^{\dagger}(0) = \frac{1}{z^{2h}} \left(1 + \frac{2h}{c} z^2 T(0) + \ldots \right) + \ldots , \tag{2.3}$$

where $A(z)$ denotes in the usual way the 'chiral (holomorphic) part' of the operator $A(z, \bar{z})$.

This result, well known and general, cannot hold true in a CFT with vanishing central charge. Indeed, a direct limit $c \to 0$ in (2.3) is not possible. We call the $1/c$ divergence in (2.3) a $c \to 0$ catastrophe.

To understand one way[l] how this catastrophe can get resolved [29], let us first consider the following example. Take a combination of two non-

[j] Operators with vanishing two-point function appear naturally in nonunitary CFTs as members of a logarithmic pair (see e.g. Eq. (2.18) below), and may perhaps be best discussed within this framework.

[k] We denote by $A^{\dagger}(z, \bar{z})$ the operator which is conjugate (more generally: 'dual') to $A(z, \bar{z})$, i.e. the one with the property that the OPE of A with $A^{\dagger}$ contains the identity operator. Our notation is understood to include, of course, the special case where $A^{\dagger} = A$.

[l] There are two more ways in which the $c = 0$ catastrophe can be resolved [29,30]: (i) by operators with vanishing two-point functions which may often naturally be thought of as members of a logarithmic pair [30], or (ii) by operators with vanishing conformal weight (to be discussed in Subsection (3.2) below).

interacting CFTs, one with central charge b, and one with central charge $-b$. We call their respective stress-energy tensors $T_b(z)$ and $T_{-b}(z)$ which satisfy the well known OPEs

$$T_b(z)T_b(0) = \frac{b/2}{z^4} + \frac{2T_b(0)}{z^2} + \frac{T'_b(0)}{z} + ... , \qquad (\text{and } \ b \to -b) \tag{2.4}$$

where prime denotes the derivative $\partial/\partial z$. The total stress-energy tensor is $T(z) = T_b(z) + T_{-b}(z)$ and the total central charge $c = b + (-b) = 0$.

A primary operator $A(z)$ of such a factorized theory would also be a product of two operators, one in the theory with positive central charge, and the other in the opposite theory. The OPE of such an operator with its conjugate can easily be found from (2.3),

$$A(z)A^\dagger(0) = \frac{1}{z^{2h}}\left(1 + \frac{h}{b}\, z^2 \left(T_b - T_{-b}\right) + ...\right) + ...$$

The problem of $c \to 0$ is now resolved, but the resolution did not come for free. We now have to introduce a new field

$$t(z) \equiv T_b(z) - T_{-b}(z) \tag{2.5}$$

with conformal weight = 2, which is different from the stress-energy tensor $T(z) = T_b(z) + T_{-b}(z)$ of the system. This field will now always appear in such OPEs in the form

$$A(z)A^\dagger(0) = \frac{1}{z^{2h}}\left(1 + \frac{h}{b}\, z^2 t(z) + ...\right) + ... \tag{2.6}$$

Continuing with our factorized theory, all OPEs between the fields $T(z)$ and $t(z)$ are easily computed from those of the factors given in (2.4),

$$T(z)T(0) = \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + ... , \tag{2.7}$$

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0)}{z^2} + \frac{t'(0)}{z} + ... , \tag{2.8}$$

$$t(z)t(0) = \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + ... \tag{2.9}$$

Note that the first equation reminds us of the fact that at central charge $c = 0$ the stress tensor $T(z)$ is a primary field with vanishing two-point function.

We would now like to generalize this analysis to theories which no longer factorize into two non-interacting theories with equal and opposite central charges. Based on our discussion to be given below, we suggest that in any

$c = 0$ CFT a field of conformal weight two, which we also denote again by $t(z)$, appears and that it enters the OPEs of primary operators with nonvanishing two-point function as in (2.6), thus resolving the $c \to 0$ catastrophe. It follows from (2.6) that $L_2 t = b$ (see e.g. (B.10) and (B.56) of Appendix B), which fixes the leading term in the OPE of $T(z)$ with $t(0)$ to be:

$$T(z)\, t(0) = \frac{b}{z^4} + \dots \, . \tag{2.10}$$

However, as far as the next order terms in this OPE are concerned, they may or may not coincide with the expansion given in (2.8). This is discussed in depth below in (2.20).

A relatively large class of nontrivial theories realizing the ('nongeneralized') OPEs (2.6), (2.8), (2.9) are affine (or: Kač–Moody) current algebras with supergroup (or: 'Lie superalgebra') symmetry, having central charge $c = 0$. One can show that a pair of *chiral* fields $t(z, \bar{z}) = t(z)$ and $\bar{t}(z, \bar{z}) = \bar{t}(\bar{z})$ with the properties discussed above, always appears in these theories [29]. These can be found as expressions quadratic in (Noether) currents, and transform under the supergroup symmetry as the 'top component' of an indecomposable multiplet of stress-energy tensors (we will discuss such multiplets in more detail in Section (5) below). The field $t(z)$ appears on the right-hand side of various OPEs such as e.g. (2.6), and obeys (2.8), (2.9). The number 'b' becomes a property of the particular affine (Kač–Moody) current algebra.

However observe that, if $t(z)$ satisfies Eqs. (2.6), (2.8), and (2.9), the algebra which T and t form becomes trivial, in the following sense. Indeed, by reversing the arguments given above, we can choose

$$T_b = (T + t)/2, \qquad T_{-b} = (T - t)/2 \tag{2.11}$$

to re-diagonalize these equations and bring them into the form of two independent (commuting) Virasoro algebras, with central charges b and $-b$, respectively. From this point of view, affine (Kač–Moody) Lie-superalgebras with $c = 0$ are nothing but tensor products of two non-interacting CFTs with equal and opposite central charges.

Quite remarkably, however, the OPEs (2.6), (2.8) and (2.9) are but a special case of a *more general set of OPEs* at $c = 0$, to be given in (2.16), (2.21) and (2.24) below. We will now proceed to study theories with this more general form of OPE.

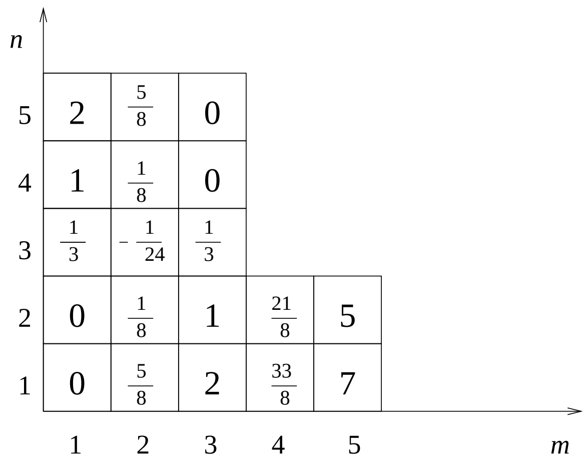

Figure 1. Some of the first few operators of the Kač table at $c = 0$

2.2. *Logarithmic partner $t(z)$ of the stress tensor $T(z)$*

A special set of primary operators, the so-called Kač-degenerate operators, have conformal weights which lie on a two-dimensional grid, usually referred to as the Kač table. It is well known that in conventional CFTs chiral (=holomorphic) correlation functions involving at least one such 'Kač-degenerate' operator satisfy [m] certain differential equations [1]. Solving such differential equations for the (chiral) four-point functions (conformal blocks), provides a way to find the OPEs of primary operators. For further reference we provide in Fig. 1 a list of the first few operators of the Kač table at $c = 0$.

Moreover, it is well known [1] that, due to global conformal invariance, the (chiral) four-point function of a primary operator [n] can be expressed in terms of a single function $F(x)$,

$$\langle A(z_1)A(z_2)A(z_3)A(z_4) \rangle = \frac{1}{(z_1 - z_2)^{2h}(z_3 - z_4)^{2h}} F(x) \,, \tag{2.12}$$

where x denotes a cross-ratio

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \,. \tag{2.13}$$

Consider the ordinary differential equation for the function $F(x)$, associ-

[m] Even though this is certainly the case in 'conventional' CFTs (as opposed, e.g., to $c = 0$ theories), as discussed in Ref. [1], this issue is, as we will see, more delicate for $c = 0$ theories; see the discussion following (3.10) below.

[n] For simplicity of presentation we have chosen here all four operators to be equal and $A^\dagger = A$.

ated with an operator A belonging to the Kač table.

In conventional CFTs (which have $c \neq 0$), there is one solution of that equation which is of the form

$$F(x) = 1 + \alpha_0 \; x^2 + ... \; , \tag{2.14}$$

(with some constant α_0) corresponding to the OPE (2.3). The function $F(x)$ with this expansion is usually referred to as the *identity conformal block* of the chiral four-point function given in (2.12).

The situation at $c = 0$ is far more complex, however. By investigating the corresponding differential equation, it can be directly verified that for all the operators from the first two rows or from the first column of the Kač table in Fig. 1 (except for those with vanishing conformal weight, discussed separately below), the small-x behavior of the identity conformal block is

$$F(x) = 1 + \alpha \; x^2 \log(x) + ... \tag{2.15}$$

in contrast to (2.14). It turns out that the other operators of the Kač table, which lie deeper in its interior (i.e. beyond the first two rows or the first column), have even more complicated identity conformal blocks [31]. We will not consider them in this paper, however.

The appearance of logarithms in a correlation function at a critical point, as on the right-hand side of (2.15), is characteristic of theories with so-called logarithmic operators [32]. In this particular case, the relevant logarithmic operator has conformal weight two, the same weight as that of the stress tensor $T(z)$. Based on these considerations we are led to suggest the following contribution to the identity operator appearing in the OPE between any two primary operators with nonvanishing two-point function,[o]

$$A(z)A^{\dagger}(0) = \frac{1}{z^{2h}} \left(1 + \frac{h}{b} z^2 \left[t(0) + \log(z) T(0) \right] + \ldots \right) + \text{other primaries}, \tag{2.16}$$

where the ellipsis denotes higher descendants of the identity operator, and '*other primaries*' denotes contributions to this OPE from operators other than the identity. This OPE generalizes Eq. (2.6). (For more details about the structure of this OPE see Sections (B.1) and (B.6) of Appendix B.)

In order to understand why an OPE of this kind would give rise to the logarithms in the conformal block (2.15), first recall [32,33] that, in general,

[o] We do not suggest that (2.16) is necessarily an appropriate OPE for Kač-table operators. Indeed, arguments given in Ref. [30] would indicate that bulk Kač-table operators for percolation and for self-avoiding polymers have vanishing two-point functions, and this would not lead to the OPE (2.16).

two (quasiprimary [1]) operators $C(z)$ and $D(z)$ of conformal weight h are said to form a *logarithmic pair*, if the dilation operator L_0 does not act diagonally, but in 'Jordan block' form, i.e.

$$L_0 C = hC\,, \qquad L_0 D = hD + C\,. \tag{2.17}$$

Global conformal invariance then enforces the following form of the two-point functions:

$$\begin{aligned} \langle\, C(z)C(0)\,\rangle &= 0\,, \\ \langle\, C(z)D(0)\,\rangle &= \frac{a_1}{z^{2h}}\,, \\ \langle\, D(z)D(0)\,\rangle &= \frac{-2a_1 \ln z + a_0}{z^{2h}}\,. \end{aligned} \tag{2.18}$$

Once the normalization of the operator C is given, the normalization of the operator D has been fixed by requiring the appearance of the *same* coefficient a_1 in the second and in the third equation. The arbitrary constant a_0 arises from the freedom to redefine the operator D by addition of C with an arbitrary coefficient, without changing the OPE's above.

The (quasiprimary) operators $T(z)$ and $t(z)$ appearing in (2.16) form precisely such a logarithmic pair, as we will see in (2.22) below. One now verifies immediately, upon identifying $C \to T, D \to t, h \to 2, a_1 \to b$, that the OPE (2.16) leads to the logarithm in the small-x expansion (2.15) of the identity conformal block of the chiral four-point function upon using (2.18) (here we consider $A^\dagger = A$ for simplicity). Furthermore, the coefficient α in (2.15) is then fixed to be

$$\alpha = \frac{h^2}{b}\,. \tag{2.19}$$

The operator $t(z)$ we introduced in this manner in (2.16) fulfills the same role as the operator $t(z)$ in (2.6) of the previous section. It makes sure the limit $c \to 0$ in (2.3) makes sense. However at the same time, it is also responsible for the logarithms in (2.15), and therefore it is a logarithmic operator. An OPE between the $c = 0$ stress tensor $T(z)$ and the new operator $t(z)$, which generalizes (2.8), and which causes $(T(z), t(z))$ to become a 'logarithmic pair', is (as we will see shortly)

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + \lambda T(0)}{z^2} + \frac{t'(0)}{z} + \dots\,, \tag{2.20}$$

where the parameter λ is arbitrary. $\lambda = 0$ corresponds to the previous, 'non-logarithmic OPE' (2.8), which implies (2.9), and this can be re-diagonalized as in the previous Section. On the other hand, nonzero λ corresponds to a new, 'logarithmic OPE', as we now explain. First, when $\lambda \neq 0$, we can redefine $t(z)$ and b by dividing by λ and arrive at [p]

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + T(0)}{z^2} + \frac{t'(0)}{z} + \dots \,. \tag{2.21}$$

This OPE generalizes the OPE (2.8) derived for the special (factorized) situation considered in the previous Section.

The OPE (2.21) fixes the following set of (holomorphic) two-point correlation functions (compare with (2.18), recalling that the normalization of the stress-tensor $T(z)$ is fixed),

$$\begin{aligned} \langle\, T(z)T(0)\,\rangle &= 0\,, \\ \langle\, T(z)t(0)\,\rangle &= \frac{b}{z^4}\,, \\ \langle\, t(z)t(0)\,\rangle &= \frac{-2b\ln z + \theta}{z^4}\,, \quad (\theta \text{ is a constant})\,. \end{aligned} \tag{2.22}$$

The first of these equations comes from the OPE (2.7) which remains unchanged, independent of whether $t(z)$ satisfies the previously discussed 'non-logarithmic' OPE Eq. (2.8), or its 'logarithmic' generalization (2.21) which we are currently considering. The second equation follows directly from the OPE (2.21), while the third equation can be computed by imposing global conformal invariance on $\langle\, t(z)t(0)\,\rangle$ (see Appendix A for more details). Hence, the 'logarithmic OPE' in (2.21) is directly responsible for the logarithm appearing in the third of (2.22), and thereby, for the logarithm in the conformal block (2.15).

As mentioned above, the constant θ remains undetermined. The occurrence of such undetermined constants is common in the theory of logarithmic operators [32, 33], and is related to the fact that $t(z)$ can be redefined as in

$$t(z) \to t(z) + \gamma T(z) \tag{2.23}$$

with an arbitrary coefficient γ. This redefinition does not affect any of the OPEs discussed in this Section.

With a *logarithmic* operator $t(z)$ satisfying (2.21) the theory is no longer equivalent to two commuting Virasoro algebras at $c \neq 0$ (the last of (2.22)

[p] The various terms are easy to understand. The leading term is fixed by the considerations of (2.10), recalling that both OPEs, (2.6) and (2.16), imply $L_2 t = b$ (Appendix B, (B.10) and (B.56)). There is no $1/z^3$ term because t is quasiprimary. The next order, $1/z^2$-term is fixed by the 'logarithmic' condition (2.17), and by the conformal weight $h = 2$ of $t(z)$.

is an obstruction to the diagonalization performed in (2.11)). On the other hand, (2.21) is the most general OPE which a conformal weight-two (quasiprimary) operator $t(z)$ with $L_2 t = b$ can satisfy (recall the footnote preceeding Eq. (2.21)). Therefore, we postulate that (2.21) is realized in all CFTs with central charge $c = 0$, except for those which simply factorize as in Subsection (2.1).

Finally, once the OPE (2.21) between t and T is known, it is possible to construct (the contribution of the identity operator to) the OPE of t with itself, which generalizes (2.9). We can do this by taking the most singular term of this OPE from the correlation function $\langle\, t(z)t(0)\,\rangle$ computed in (2.22). This function contains an ambiguity related to the possibility to redefine $t(z)$ according to (2.23). In what follows, we fix this ambiguity by setting $\theta = 0$ in (2.22). The most singular term entails all other terms of the OPE, which becomes

$$t(z)t(0) = -\frac{2b\log(z)}{z^4} + \frac{t(0)\,[1-4\log(z)] - T(0)\left[\log(z) + 2\log^2(z)\right]}{z^2} + \frac{t'(0)\,[1-4\log(z)] - T'(0)\left[\log(z) + 2\log^2(z)\right]}{2z} + \dots, \quad (2.24)$$

where the ellipsis denotes higher order terms, as well as contributions from primary operators other than the identity operator. The technique for reconstructing entire OPEs such as (2.24) from their most singular terms is well known and is described in Ref. [1], although its application to the theory with logarithms has not often been discussed in the literature. Briefly, it consists of the following steps (more details can be found in Appendix B, especially Sections B.4 and B.8). First we have to derive the commutation relations between the Virasoro generators L_n and t, from (2.21). This yields

$$[L_n, t(z)] = \left(z^{n+1}\frac{d}{dz} + 2(n+1)z^n\right)t(z) + (n+1)z^n T(z) + \frac{b}{3!}(n^3-n)z^{n-2}\,. \quad (2.25)$$

Then we apply L_n with $n \geq 0$ to (2.24). On one hand, L_n can be applied to the right-hand side of (2.24) directly. On the other hand, we can use

$$[L_n, t(z)t(0)] = [L_n, t(z)]t(0) + t(z)[L_n, t(0)]$$

on the left-hand side, substitute (2.25) into this expression, and use the OPEs $t(z)t(0)$ and $T(z)t(0)$ to find relationships between various terms in (2.24). Ultimately, this allows us to deduce, order by order in z, all the terms in (2.24) from its most singular term.

Equation (2.24) fixes the OPE $t(z)t(0)$ up to contributions of other primary operators. $T(z)$ is a primary operator at $c = 0$ and, since it already appears in Eq. (2.24), we can expect on general grounds that there could be a 'stand alone' contribution of the conformal block of the stress tensor to this OPE. This amounts to

$$\frac{2a\ T(0)}{z^2} + \frac{a\ T'(0)}{z} + ... \tag{2.26}$$

being added to the right-hand side of Eq. (2.24) where a is an arbitrary coefficient. This was recently stressed by I. Kogan and A. Nichols [34]. Additionally, one could imagine that in specific $c = 0$ CFTs which realize the logarithmic operator $t(z)$, there could be contributions from other primary operators on the right-hand side of Eq. (2.24) (as already mentioned). These, however, will play no further role in this paper, until we arrive at Subsection (5.2).

As we saw in the Introduction, in certain cases it is possible to think of a $c = 0$ CFT as a limit of a continuous set of CFTs parametrized by a parameter n, defined in an interval containing $n = 0$, where the central charge $c(n) \neq 0$ if $n \neq 0$, and $c(0) = 0$. In that case, we could ask how a partner $t(z)$ of the stress tensor can appear in the limit $n \to 0$, while it is definitely not present in the theory at $n \neq 0$. The answer to that question was given by J. L. Cardy in Ref. [30, 35].

We summarize by saying that the OPEs (2.21) and (2.24), together with the OPE of the stress tensor $T(z)$ with itself (which is unmodified, and as in (2.7)) constitute the fundamental equations of a CFT at central charge $c = 0$, which does not factorize as in Subsection (2.1).

3. Implications of the logarithmic $t(z)$ and corresponding b on $c = 0$ Kač-table operators with nonvanishing two-point functions

In this section we study the implications of the 'anomaly' number b for null-vectors, associated with primary operators which have conformal weights listed in the $c = 0$ Kač table, and nonvanishing two-point functions.[q] For ordinary (e.g. unitary) CFTs there is no issue, because the null-vectors are known to vanish when inserted into any (chiral) correlation function with other operators [1]. This step is no longer guaranteed to be valid for the nonunitary theories discussed here. In the first part, Subsection 3.1, we

[q] As already mentioned, operators with vanishing two-point function may often naturally be viewed as members of a logarithmic pair (see e.g. Eq. (2.18) above).

demonstrate how the 'anomaly' number b controls this issue, for primary operators with certain nonvanishing Kač-table weights (and non-vanishing two-point functions). In the second part, Subsection 3.2, we discuss similar statements for Kač-table operators with vanishing conformal weight (and nonvanishing two-point functions). A convenient tool used in both subsections to address these questions in a purely algebraic way, is a (partial) extension of the Virasoro algebra by suitably defined Laurent modes of the logarithmic partner $t(z)$ of the stress tensor.

3.1. *Operators with nonvanishing dimensions*

In view of the relation (2.19)

$$\alpha = \frac{h^2}{b}$$

it may appear, at first sight, that a separate, and possibly different value of the parameter b could be associated with different $c = 0$ primary operators $A(z)$. This would mean, that a given theory would contain two (or more) different values of b, say $b \neq b'$. But this would imply that there would exist two (or more) different operators $t(z)$, say $t_b(z)$ and $t_{b'}(z)$, each obeying the OPE (2.21) with the coefficient of the corresponding $1/z^4$ term equal to b and b', respectively. Then it is not difficult to see that the correlator $\langle\, t_b(z)t_{b'}(0)\,\rangle$ violates global conformal invariance. (Details are given in Appendix A.) This means that different values of b cannot coexist in the same theory, and that b is a characteristic of any $c = 0$ CFT. Therefore, the question arises what value the number b takes in a given theory, and what conditions b imposes on the properties of the $c = 0$ CFT. This is the question we address in this Section.

We begin by considering a $c = 0$ theory containing in its operator content one or more primary operators whose conformal weight appears in the Kač table. If we were to assume that the corresponding null-vector, implied by the Kač-table conformal weight, can be set itself to zero,[r] then any (chiral) four-point function (conformal block) involving this operator would satisfy a differential equation. For any primary operator with Kač-table conformal weight (and nonvanishing two-point function) it would hence be possible to extract the coefficient α appearing in (2.15) from the solution of the corresponding differential equations for $F(x)$ (see (2.12)). Therefore, in view of (2.19), this associates a value of b with any such Kač-table operator.

[r] See (3.10) below for an example, and a more in-depth discussion of this issue.

In the following, we will obtain a purely algebraic way of associating a number b with operators in the $c = 0$ Kač table which have nonvanishing two-point function, without explicitly referring to the corresponding differential equation, or its solutions. Interestingly, we find that different values of b appear in the $c = 0$ Kač table. For example, operators in the first two rows of the Kač table have $b = +5/6$, whereas operators in the first column have $b = -5/8$ (see (3.15) below).

Since the value of the number b is unique in a given theory (as per our discussion at the beginning of this section), this implies that only those Kač-table operators which have a given fixed value of b can give rise to differential equations in a given theory.

In order to arrive at our algebraic determination of b for operators with conformal weights contained in the $c = 0$ Kač table, we will first establish the OPE between $t(z)$, and an arbitrary primary operator $A_h(z)$ of conformal weight $h \neq 0$ and nonvanishing two-point function.

We start by determining the three-point correlation functions involving these operators. It is well known that the three-point correlation functions are completely determined by global conformal invariance. By imposing global conformal invariance on $\langle\, t(z)A_h(w_1)A_h(w_2)\,\rangle$ one readily finds

$$\langle\, t(z)A_h(w_1)A_h(w_2)\,\rangle = \frac{h\log\left(\frac{w_1-w_2}{(z-w_1)(z-w_2)}\right)+\Delta}{(z-w_1)^2(z-w_2)^2(w_1-w_2)^{2h-2}}\,. \qquad (3.1)$$

The coefficient Δ is arbitrary and is not fixed by conformal invariance. In what follows we set $\Delta = 0$, which amounts to redefining $t(z)$ as in (2.23) in a suitable way. (Notice that this would not be possible for operators A_h with vanishing conformal weight, $h = 0$. Therefore we consider for now only operators A_h with nonvanishing conformal weight.) Now consider expanding the three-point function (3.1) for small $(z_1 - w_1)$. One immediately sees that the term multiplying $\log(z_1 - w_1)$ is precisely equal to $< T(z)A_h(w_1)A_h(w_2) >$. Moreover, an additional power series in $(z - w_1)$ appears which does not multiply $\log(z_1 - w_1)$. All this is consistent with the following OPE,

$$t(z)A_h(0) = -T(z)A_h(0)\log(z) + \sum_{n=0}^{+\infty} \ell_{-n}A_h(0)z^{n-2} \quad + \quad \Re(z)\,, \qquad (3.2)$$

where potential noninteger powers of the variable z (but no logarithms) are collected in a ‘remainder’ denoted by $\Re(z)$. This can be viewed as a definition of the operators $\ell_{-n}A_h(0)$. Furthermore, if the operator A_h is replaced by a more general (e.g. nonprimary) operator, negative powers of the index n may appear in the OPE (3.2).

Alternatively, the action of the operators ℓ_n on the operator $A_h(0)$ can be computed as usual by contour integration from (3.2),

$$\ell_n = \hat{\mathbf{P}}_h \left[\oint \frac{dz}{2\pi i} \Big(t(z) + \log(z) T(z) \Big) z^{n+1} \right] \hat{\mathbf{P}}_h, \qquad (n \in \mathbf{Z}), \tag{3.3}$$

where $\hat{\mathbf{P}}_h$ is the projection operator on all states of the Hilbert space of the CFT whose conformal weights differ from the weight h of the operator $A_h(0)$ by an integer (this projection operator commutes with the operator $T(z)$; see e.g. (B.39)).

Formula (3.3), together with the definition of the conventional Virasoro generators

$$L_n = \oint \frac{dz}{2\pi i} \; T(z) z^{n+1}, \tag{3.4}$$

allows us to find the commutation relation $[\ell_n, L_m]$ in the familiar manner from the OPE (2.21). The result is

$$[\ell_n, L_m] = +(n-m)\ell_{n+m} - m L_{n+m} + \frac{b}{6} n(n^2-1)\delta_{n+m,0} \; . \tag{3.5}$$

It constitutes a (partial) generalization of the commutation relations of the Virasoro algebra, which are given at $c = 0$ by

$$[L_n, L_m] = (n-m) L_{n+m} \, . \tag{3.6}$$

We now claim that

$$\ell_n A_h(0) = 0 \, , \qquad \text{for all} \;\; n \geq 0 \tag{3.7}$$

for primary operators A_h with nonvanishing two-point function. First, it clearly follows from the OPE (3.2) that this is true for $n > 0$. Moreover, we can choose a definition of the operator $t(z)$ so that $\ell_0 A_h(0) = 0$; indeed, since $t(z)$ is defined up to addition of $T(z)$ as in (2.23), ℓ_n is also defined up to addition of L_n, as in (2.23). By adding L_0 to ℓ_0 with a suitable coefficient, we can always[s] make $\ell_0 A_h(0)$ vanish as long as $h = L_0 A_h(0) \neq 0$. From now on we assume that we have chosen $t(z)$ in this way, while we consider the operator $A_h(z)$.

[s] A different such 'subtraction' will typically have to be performed for each operator $A_h(z)$ separately. This will not affect the arguments given below.

Using the newly derived commutators (3.5), the ordinary Virasoro algebra (3.6), as well as (3.7), one finds that

$$L_1\left(\ell_{-1}-\frac{1}{2}L_{-1}\right)A_h(0)=0. \tag{3.8}$$

Therefore, $(\ell_{-1}-\frac{1}{2}L_{-1})|A_h\rangle$ can be called a *null-vector*, following Ref. [1]. Indeed, from now on we will set this null-vector to zero, i.e.

$$\ell_{-1}A_h(0)=\frac{1}{2}L_{-1}A_h(0), \tag{3.9}$$

which is easily seen to be consistent with the correlation function (3.1). Moreover, this is also consistent with the general constraints of conformal symmetry imposed on the OPE (3.2) (see e.g. Section B.7 of Appendix B).

Any primary operator whose conformal weight appears in the Kač table has descendants which are themselves null-vectors. Specifically, this means that there exist states $|\xi\rangle$, constructed by applying Virasoro lowering operators L_{-m} $(m>0)$, to the primary state $A_h|0\rangle$, so that $L_n|\xi\rangle=0$, for all $n>0$. However, now we have a (partial) extension of the Virasoro algebra in hand, generated by L_n and ℓ_n. Hence it is natural to ask if the null-vectors are annihilated by ℓ_n as well as by L_n. Consider for example the primary operator with conformal weight $h_{(2,1)}=\frac{5}{8}$ which is contained in the $c=0$ Kač table. We assume it has nonvanishing two-point function. Its (Virasoro) null-vector is

$$\left(L_{-2}-\frac{2}{3}L_{-1}L_{-1}\right)A_{\frac{5}{8}}(0) \tag{3.10}$$

which means that this expression is annihilated by L_n with $n>0$. We are interested in knowing if we are allowed to set the operator appearing in (3.10) *itself to zero*, when it occurs in *any* correlation function with other operators. This is important to know, because if true it would give rise [1] to a differential equation satisfied by any conformal block involving the Kač-table operator $A_{5/8}$. In view of the nonunitarity of the present theory, it is not obvious that (3.10) itself can be set to zero. (See the paragraph below, containing (3.13), for a related example.) Now we observe that by applying the operator ℓ_2 to (3.10) and by using (3.5) one finds

$$\ell_{+2}\left(L_{-2}-\frac{2}{3}L_{-1}L_{-1}\right)A_{5/8}(0)=\big(b-5/6\big)A_{5/8}(0) \tag{3.11}$$

which vanishes only if $b=\frac{5}{6}$. Furthermore, when applying the operator ℓ_1

to (3.10) we arrive, irrespective of the value of "b", at

$$\left(\ell_{-1} - \frac{1}{2}L_{-1}\right) A_{\frac{5}{8}}(0), \tag{3.12}$$

which, as we have already established in (3.9), vanishes for all primary operators $A(z)$ with nonvanishing conformal weight. (Applying ℓ_n with $n \geq 3$ is easily seen to always annihilate (3.10).)

Hence we arrive at the important conclusion that *if* the descendant (3.10) of a Kač-table operator may be set to zero in any correlation function with other operators (which, we emphasize again, implies [1] the validity of the corresponding differential equation involving this operator), then ℓ_2 and ℓ_1 applied to it must also vanish,[t] and this does not happen unless $b = \frac{5}{6}$. This is a necessary condition determining the value of b by simple algebra, without solving the differential equations ensuing from the null-vector condition. On the other hand, if we took the route described at the beginning of this subsection, i.e. if we solved the differential equation for the four-point function of the operator $A_{\frac{5}{8}}(z)$ (assuming it had a nonvanishing two-point function), found the coefficient α defined in (2.15), and determined b via (2.19), it would also be $\frac{5}{6}$. Therefore, the above steps establish a purely algebraic way to determine for each Kač-table operator with nonvanishing two-point function a value of b, for which the null-vector (such as e.g. (3.10)) may be set to zero. This, in turn, gives rise to the ensuing differential equation.

The notion of a nonvanishing null-vector may be unfamiliar. To illustrate it, let us give a related, but simpler example: at $c = 0$ the stress tensor, which is a descendant of the identity operator (e.g. (B.39)), is itself primary,

$$L_{+2}L_{-2}\mathbf{1}(0) = 0\,, \qquad L_{+1}L_{-2}\mathbf{1}(0) = 0\,. \tag{3.13}$$

The stress tensor hence represents a null-vector of the identity operator at level two. But clearly, the stress tensor does not vanish, even though its two-point function does. And indeed, in analogy with (3.11), we obtain from (3.5)

$$\ell_{+2}L_{-2}\mathbf{1}(0) = b\ \mathbf{1}(0) \tag{3.14}$$

which does not vanish (unless $b = 0$ in this case).

While we lack a general result for b based on the above method for *all* operators in the $c = 0$ Kač table, we have repeated this procedure for many

[t] Consider e.g. a correlation function involving $t(z)$, which is known to appear in OPEs between primary operators, such as (2.16); see Appendix C for further elaboration.

of the operators of the Kač table and found the following pattern [36]:

$$\text{for } A_{(k,1)} \ \&\ A_{(k,2)},\ k > 1 \quad \text{(first two rows)}: \qquad b = +\frac{5}{6},$$
$$\text{for } A_{(1,k)},\ k > 2 \quad \text{(first column)}: \qquad b = -\frac{5}{8}. \tag{3.15}$$

Here $A_{(m,n)}$ denotes the operator located in position (m, n) of the Kač table of Fig. 1.

In view of the uniqueness of the number b in any given theory, the appearance of *different* values for b in (3.15) has important consequences. It means that, in a given $c = 0$ theory, only certain subsets of primary operators with conformal weights given by the Kač-table have null-vectors which vanish identically (implying that the corresponding conformal blocks satisfy differential equations). To be entirely clear, but at the risk of being repetitive, let us spell this out once more in detail (all Kac-table operators mentioned below are assumed to have nonvanishing two-point function). Take for example the operator $A_{(2,1)}$ of conformal weight $\frac{5}{8}$, which we previously denoted as $A_{\frac{5}{8}}$. Conformal blocks involving this operator satisfy the (second order) differential equation associated with the null-vector of $A_{(2,1)}$ at the second level *only if* $b = \frac{5}{6}$. On the other hand, take the operator $A_{(1,3)}$ with conformal weight $\frac{1}{3}$. Conformal blocks involving this operator satisfy the (third order) differential equation associated with the null-vector of $A_{(1,3)}$ on the third level *only if* $b = -\frac{5}{8}$. Therefore, these two operators cannot give rise to the corresponding differential equations simultaneously in the same theory. That does not mean that primary operators with 'wrong' conformal weights are necessarily forbidden in the same theory. But it means that for 'wrong' operators the null-vectors cannot be set to zero, which implies that their correlation functions would not satisfy the differential equations which would otherwise follow from these null-vectors (such as (3.10)) according to the rules of Ref. [1]. For example, if $b = \frac{5}{6}$, only the (second order) differential equation associated with the null-vector of $A_{(2,1)}$ at the second level can be valid. But the (third order) differential equation associated with the null-vector of $A_{(1,3)}$ on the third level would not be valid. In other words, a conformal block involving $A_{(1,3)}$ would not satisfy this differential equation. Conversely, if $b = -\frac{5}{8}$, only the (third order) differential equation associated with the null-vector of $A_{(1,3)}$ on the third level can be valid. But the (second order) differential equation associated with the null-vector of $A_{(2,1)}$ at the second level would not be valid. In other words, a conformal block involving $A_{(2,1)}$ would not satisfy this differential equation. Finally, if b is not equal to either of these numbers, neither of these differential equations will be satisfied by the conformal blocks involving these operators. (An explicit example illustrating these issues for the Kač-table operators $A_{3,1}$ and $A_{1,5}$ of conformal weight two can be found in (5.7) below.)

This concludes our discussion of the operators with nonvanishing conformal weights.

3.2. *Operators with vanishing dimension*

Let us concentrate now on the operator $A_{(1,2)}$, appearing in position $(1,2)$ of the Kač table in Fig. 1. A remarkable feature of this operator is that its conformal weight vanishes at central charge $c = 0$, even though the operator itself is different from the identity operator. It is well known that this can happen in nonunitary theories, and the operator $A_{(1,2)}$ plays a prominent role in the theory of percolation [42].

Consider first the correlation function on the left-hand side of (3.1) where $A(z)$ is now the zero-dimensional operator with $h = 0$. In what follows, we denote this operator by $O(z)$, and we assume that it has nonvanishing two-point function. The right-hand side of (3.1) has been obtained from global conformal invariance alone, and is therefore certainly also valid when the conformal weight $h = 0$. In this case it reduces to

$$\langle\, t(z)O(w_1)O(w_2)\,\rangle = \frac{\Delta(w_1-w_2)^2}{(z-w_1)^2(z-z_2)^2}\,. \tag{3.16}$$

A difference between (3.16) and (3.1) for the operator with nonvanishing conformal weight is the absence of the logarithms. Also, it is no longer possible to set Δ to zero by employing (2.23).

Now consider how the OPE (2.16) changes when $h = 0$. It becomes

$$O(z)O(0) = 1 + Cz^2\big(T(z) + ...\big) + ... \tag{3.17}$$

(as one might have expected from (2.3)). Notice the logarithms no longer appear, and a contribution of the stress-energy tensor conformal block appears, with an arbitrary coefficient C. The associativity of the correlation function $\langle\, O(z_1)O(z_2)A_h(z_3)A_h(z_4)\,\rangle$, where A_h is an arbitrary primary operator with conformal weight h, requires $C = \frac{\Delta}{b}$.

Finally, consider how the OPE (3.2) changes. It becomes

$$t(z)O(0) = -(1-\epsilon)T(z)O(0)\log(z) + \sum_n \ell_n O(0) z^{-n-2} + ...\,, \tag{3.18}$$

where ϵ is a new constant which cannot be determined from conformal invariance alone.

It follows from (3.16) that $\ell_0 O(0) = \Delta O(0)$, in contrast to the case of operators with nonvanishing conformal weight $A(z)$, for which we can arrange for $\ell_0 A(0) = 0$ by a redefinition as in (2.23). From (3.18) something even more drastic follows: the commutation relations of ℓ_n and L_m, when they act on the zero-dimensional operator O, change from (3.5) to [u]

$$[L_n, \ell_m] = \frac{b}{6}(n^3 - n)\delta_{n+m,0} + (n-m)\ell_{n+m} + (n+\epsilon)\, L_{n+m}\,. \tag{3.19}$$

Let us try to use the commutators (3.19) to determine if the vanishing of the null-vector of $A_{(1,2)}$ at the second level imposes any constraints on the parameters b, Δ and ϵ. In this case the null-vector is $\left[L_{-2} - \frac{3}{2}L_{-1}L_{-1}\right] A_{(1,2)}$. By applying ℓ_2 and ℓ_1 (as before), we find that the following conditions need to be satisfied, if the null-vector itself can be set to zero,

$$\begin{aligned} b &= 5\Delta\,, \\ \Delta &= \frac{-5+7\epsilon}{12}\ . \end{aligned} \tag{3.20}$$

Assuming various values of ϵ yields the following values of b:

$$\begin{aligned} \epsilon = 0 \ &\rightarrow\ b = -\frac{25}{12}\,, \\ \epsilon = \frac{1}{2} \ &\rightarrow\ b = -\frac{5}{8}\,, \\ \epsilon = 1 \ &\rightarrow\ b = \frac{5}{6}\,. \end{aligned} \tag{3.21}$$

Quite independently, the differential equation associated which the operator $A_{(1,2)}$ allows us to determine Δ. Indeed, let us calculate the (chiral) four-point correlation function $\left\langle A_{(1,2)}(z_1)A_{(1,2)}(z_2)A_h(z_3)A_h(z_4)\right\rangle$ where A_h is an arbitrary primary operator with conformal weight h. Writing the second order differential equation which follows from setting to zero the null-vector of $A_{(1,2)}$ at the second level, we find an identity conformal block of the following form,

$$\left\langle A_{(1,2)}(z_1)A_{(1,2)}(z_2)A_h(z_3)A_h(z_4)\right\rangle = \frac{1}{(z_3-z_4)^{2h}}\left(1 + \frac{h}{5}x^2 + ...\right), \tag{3.22}$$

where, as before,

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}\ .$$

[u] A situation, where the commutation relations depend on the operators on which they act is encountered in other CFTs. An example of this is the parafermion CFT [43].

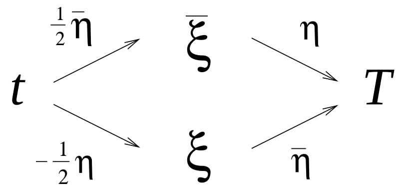

Figure 2. The multiplet of fields of conformal weight-two for the supergroup U(1|1). η and $\eta^\dagger$ are two fermionic generators of this group. (In the figure, $\eta^\dagger$ is denoted by $\bar\eta$, and $\xi^\dagger$ is denoted by $\bar\xi$.)

Matching the coefficient $\frac{1}{5}$ with the OPEs (2.16) and (3.17) we find $C = \frac{1}{5}$ and consequently $\Delta = \frac{b}{5}$. This is consistent with the purely algebraic results obtained above. The value of b cannot be determined in this way, however. This ends our discussion of operators with vanishing conformal weight.

4. Critical Disordered Systems

Consider a generic disordered system where the disorder average can be performed using the supersymmetry (SUSY) method, resulting in an action such as that given in (1.9). A theory of this kind is always invariant with respect to isotopic supersymmetry ('supergroup') transformations. For example, the action given in (1.9) is invariant under superunitary rotations

$$\begin{pmatrix} \phi' \\ \psi' \end{pmatrix} = U \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{4.1}$$

where U is a superunitary matrix. According to Ref. [29] the stress tensor of such systems is always a member of a certain indecomposable SUSY multiplet. The number of fields in this multiplet depends on the symmetry group of the system. Any such theory must, however, at least be invariant under a minimal U(1|1) SUSY, giving rise to a 4-dimensional (indecomposable) multiplet of stress tensors denoted by $T(z)$, $t(z,\bar z)$, $\xi(z,\bar z)$, $\xi^\dagger(z,\bar z)$ in Ref. [29]. All four fields must have conformal weights $(h,\bar h) = (2,0)$. This multiplet, together with the action of the U(1|1) generators denoted (as in [29]) by $\eta, \eta^\dagger, j, J$, satisfying the relations

$$[j,\eta] = -2\eta, \quad \left[j,\eta^\dagger\right] = +2\eta^\dagger, \quad \{\eta,\eta^\dagger\} = J, \tag{4.2}$$

is depicted in Fig. 2. (The operators $(t, \xi^\dagger, \xi, T)$ transform in the same way as $(j, \eta^\dagger, \eta, J)$.) If the SUSY is larger than this minimal $U(1|1)$, then this multiplet will in general contain more fields, but the above four will always be contained therein.

A special role is played by the 'top' field $t(z,\bar{z})$ of the multiplet, displayed in Fig. 2, whose OPE with the stress tensor T was argued in Ref. [29] to satisfy (cf. Eq. (2.10))

$$\langle T(z)t(w,\bar{w})\rangle = \frac{b}{(z-w)^4}\ , \tag{4.3}$$

where the parameter b counted [v] the number of effective degrees of freedom of the disordered system. Different disordered systems, all having central charge $c=0$, can be distinguished by different values of b. It was further suggested in Ref. [29] that the field $t(z,\bar{z})$ together with the stress-tensor $T(z)$ should generate a certain extension of the Virasoro algebra, via their OPE. However, the most general form of this OPE was not established.

At a critical point a disordered system will typically have, as mentioned, vanishing central charge $c=0$. According to the analysis in Section 2.1, this implies the existence of an operator with holomorphic part $t(z)$, designed to avoid the '$c\to 0$ catastrophe'. At this stage, upon comparing (4.3) with (2.10), we are forced to identify the holomorphic part of $t(z,\bar{z})$ obtained in Ref. [29] by supersymmetry methods, with $t(z)$ considered in Section 2.1 of this paper. Thus, at supersymmetric disordered critical points, a partner of the stress tensor is known to appear simply on grounds of (super-) symmetry.

An important remark has to be made at this point. We know from the analysis of Ref. [29] that in general, the operator $t(z,\bar{z})$ does not have to be holomorphic, as indicated by the notation. Although there exist theories such as Kač–Moody super current algebras, where it is certainly holomorphic (but not logarithmic, as discussed in Section 2.1), $t(z,\bar{z})$ will in general also have a nontrivial dependence on $\bar{z}$. Quite analogously, the corresponding operator $\bar{t}(z,\bar{z})$, which is related to $\bar{T}(\bar{z})$ in the same way as $t(z,\bar{z})$ is related to $T(z)$, can depend on z.

From the point of view of its $\bar{z}$ dependence, $t(z,\bar{z})$ must have dimension zero. Therefore, only two options are allowed for t. One is given by $\bar{L}_0 t = 0$. This means that either t does not depend on $\bar{z}$, or, that its $\bar{z}$-dependence arises from a weight-zero operator, different from the identity. The other option is given by

$$\bar{L}_0 t = T. \tag{4.4}$$

This was first suggested in Ref. [44]. See also Ref. [30].

[v] In a replica theory, the number b basically corresponds [27, 30, 34, 44] to the so-called 'effective central charge' $\partial/\partial n_{|n=0}\ c(n)$, where $c(n)$ is the central charge of n coupled replicas, if only one uses a slightly different normalization of the operator $t(z)$ such as in (2.20).

In this note we focus entirely on the z-dependence. The dependence of t on $\bar{z}$ would have to treated separately, and finally the holomorphic and antiholomorhpic parts will have to be put together. However, we do not address the issues of the $\bar{z}$-dependence of $t(z, \bar{z})$ here. This is an important problem which has not been properly investigated up until now, and should be a subject of future work.

5. Extended Stress Tensor Multiplet

In this section we first focus entirely on consequences of conformal symmetry, without regard to SUSY. In particular, we will simply consider a general CFT at $c = 0$ which, as discussed in Subsection 2.2 above, possesses a stress energy tensor T as well as a logarithmic partner t, satisfying the OPEs described in Eqs. (2.21) and (2.24). Such a theory may for example appear as the $n \to 0$ limit of a replicated theory of n interacting copies of fields, as discussed in the Introduction. We are going to show that the requirement that $t(z)$ is logarithmic automatically ensures the existence of extra primary conformal weight-two fields with nonvanishing two-point function. Moreover, if there are *two* such extra fields, $\xi(z)$ and $\xi^\dagger(z)$, which are *anticommuting*, then the OPEs between all four holomorphic weight-two fields $T, t, \xi, \xi^\dagger$ automatically transform covariantly under a global $U(1|1)$ supersymmetry. In other words, we are going to show that the requirements of conformal invariance, $c = 0$, and the OPE (2.21), together then imply invariance under a global supersymmetry acting on the holomorphic parts of these fields. (We emphasize that there was no mention of SUSY at the outset.) Whether or not this will in fact translate into an actual global SUSY of the full $c = 0$ theory will depend on the specific gluing of holomorphic and antiholomorphic sectors, an aspect that remains to be explored and which we are not addressing here.

5.1. *The multiplet of two dimensional operators, and SUSY*

So far, the operators with conformal weight two are $t(z)$, which is not primary according to the OPE (2.21), and $T(z)$, which is primary at $c = 0$, but has a vanishing two-point correlation function. It turns out, however, as we will now see, that these two fields do not exhaust all conformal weight-two operators we need to introduce for consistency of our theory. We also need to add conformal weight-two primary operators *with nonvanishing correlation functions.*

Consider, a (chiral) correlation function with insertions of several operators $t(z)$. Such a correlation function will not be single-valued. In order to see how the additional weight-two operators appear, it is instructive to examine the OPE between two operators $t(z)$, given in Eq. (2.24). Due to the presence of the logarithm, this OPE is clearly not single-valued as z goes around zero. A correlation function with a $t(z)t(0)$ insertion will change under such a transformation. A piece will be added to it whose small z behavior can be found by shifting all the logarithms in (2.24) by $2\pi i$.

It is possible to establish that this piece will contain a full OPE between two primary operators A with conformal weight two, and nonzero two-point correlation function. To be specific let us introduce the conformal weight-two primary field $A_2(z)$. Its OPE with itself has the following contribution from the identity operator

$$A_2(z)A_2(0) = \alpha T(z)T(0) + \frac{b}{2z^4} + \frac{t(0) + T(0)\log(z)}{z^2} + ... , \tag{5.1}$$

in agreement with (2.16). Here we fixed the normalization of A_2 in a certain way which will become useful later. We also introduced a piece in the OPE proportional to the arbitrary coefficient α. It is allowed by conformal invariance and its utility will become obvious as we go along. In the previous sections we routinely set coefficients such as α to zero by employing the redefinition (2.23), as well as the OPE (2.7) of the stress tensor. However, at this stage we have already used this redefinition once, to fix the OPE Eq. (2.24), so we cannot use it again. Notice that the three-point function (3.1) is now fixed to

$$\langle\, t(z)A_2(w_1)A_2(w_2)\,\rangle = b\,\frac{\log\left(\frac{w_1-w_2}{(z-w_1)(z-w_2)}\right) + 2\alpha}{(z-w_1)^2(z-z_2)^2(w_1-w_2)^2}\,. \tag{5.2}$$

It follows that the OPE $t(z)A_2(w)$ is fixed to be

$$t(z)A_2(0) = 2\alpha T(z)A_2(0) - T(z)A_2(0)\log(z) + \frac{A_2'(0)}{2z} + ... , \tag{5.3}$$

similar to the OPE (3.2), but again with the extra term proportional to α. So far α remains arbitrary.

Now it can be checked, by using (2.24) and (5.1), that the contribution of (the Virasoro representation of) the identity operator in the following combination of OPEs

$$t(z)t(0) + 4A_2(z)A_2(0)\log(z) - T(z)T(0)\ln^2(z) + \left(\frac{1}{2} - 4\alpha\right)T(z)T(0)\ln(z) \tag{5.4}$$

no longer contains any logarithms, to all orders in z. (A proof is presented in Subsection B.9 of Appendix B.) It therefore remains single-valued as a function of z, when inserted into any correlation function. This shows that indeed, an insertion of $t(z)t(0)$ into any (chiral) correlation function with other operators, will analytically continue to a linear combination of $t(z)t(0)$ and $A_2(z)A_2(0)$ (and $T(z)T(0)$). This is how the conformal weight-two primary fields $A_2(z)$ will appear in any theory at $c = 0$ which contains the logarithmic field $t(z)$.

We already established in the Section 3.1 that (chiral) correlation functions containing the conformal weight-two primary operator $A_2(z)$ (with nonvanishing two-point function) would satisfy the corresponding differential equation, if b is either $\frac{5}{6}$ or $-\frac{5}{8}$. In the first case, this would be the third order equation for the operator $A_{(3,1)}$, and in the second the fifth order equation for the operator $A_{(1,5)}$. In fact, the solutions to these equations for the identity conformal block of the four-point function $\langle A_2 A_2 A_2 A_2 \rangle$ can be obtained in closed form. An important feature of these solutions is that the identity conformal block $F(x)$ defined as in (2.12), (2.13), if normalized to 1 as $x \to 0$, goes to $(-x^4)$ as $x \to \infty$, and vanishes altogether when $x \to 1$ (or the opposite way around). This is only possible if there exist *two* conformal weight-two operators which are in fact *anticommuting operators*. Of course we could also consider *commuting* operators as far as the considerations leading to Eq. (5.4) are concerned, but their correlation functions would not satisfy the appropriate differential equations at the appropriate values of b.

Motivated by these considerations let us consider, from now on, the case of two conjugate *fermionic* conformal weight-two operators $\xi(z)$ and $\xi^\dagger(z)$, but we no longer require that they be Kač-operators; in particular, the parameter b can now take on arbitrary values. (Besides the fermionic nature of ξ and $\xi^\dagger$ we make no other assumptions but conformal symmetry.) Their OPE can be copied from (5.1),

$$\xi(z)\xi^\dagger(0) = \alpha T(z)T(0) + \frac{b}{2z^4} + \frac{t(0) + T(0)\log(z)}{z^2} + \ldots . \tag{5.5}$$

It is remarkable that the identity conformal block of the four-point function of these operators,

$$G = \left\langle \xi(z_1)\xi^\dagger(z_2)\xi^\dagger(z_3)\xi(z_4) \right\rangle , \tag{5.6}$$

can be obtained in closed form for arbitrary values of the parameter b by a generalization of the conformal Ward identity. (Note that unless $b = 5/6$ or $b = -5/8$, this function no longer satisfies a corresponding differential equation.) In order to see this, consider the linear combination

$$\left\langle \xi(z_1)\xi^\dagger(z_2)\xi^\dagger(z_3)\xi(z_4)\right\rangle - \frac{1}{2}\left\langle T(z_1)T(z_2)\xi^\dagger(z_3)\xi(z_4)\right\rangle \log\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}.$$

This combination is a rational function of the coordinates, as can be checked by employing the OPEs. Therefore, it can be reconstructed from its poles, as in the standard conformal Ward identity [1]. This allows us to find the correlation function G for arbitrary values of b. This correlation function must of course vanish as z_1 approaches z_4, due to the fermionic nature of the operators ξ. It turns out that it vanishes only if the parameter α in (5.5) is chosen to be $\alpha = \frac{1}{8}$. This explains why the parameter α was introduced in (5.5) in the first place. Setting $\alpha = 1/8$, we obtain the explicit result,

$$G = \frac{b}{(z_1-z_2)^4(z_3-z_4)^4}\left[\frac{(x+1)(2x^2+b(x-1)^2(1+x^2))}{4(x-1)} - \frac{x^2(1-x+x^2)\log(x)}{(x-1)^2}\right]. \tag{5.7}$$

We can check by direct substitution that (the function of the cross-ratio x, associated as in (2.12) with the function) G satisfies the third order equation for the operator $A_{(3,1)}$ if $b = \frac{5}{6}$, and that it satisfies the fifth order equation for the operator $A_{(1,5)}$ if $b = -\frac{5}{8}$. It is not completely obvious from (5.7) that $G \to 0$ when $z_1 \to z_4$ (which implies $x \to 1$), but it can be easily checked with the help of straightforward algebra.

Let us summarize briefly. We have established that $c = 0$ conformal theories with a logarithmic operator $t(z)$ must contain, in addition to the stress tensor $T(z)$ and $t(z)$, extra operators with conformal weight two but nonvanishing two-point function. For $b = \frac{5}{6}$ or $b = -\frac{5}{8}$, two *anticommuting* operators ξ and $\xi^\dagger$ are required, if the latter are to satisfy the corresponding Kač-table null-vector conditions (implying that the identity conformal block of (5.6) satisfies the 3rd-, or the 5th-order differential equation, corresponding to their respective Kač-table positions). In general, as long as these two conformal weight-two operator ξ and $\xi^\dagger$ are fermionic, the full identity conformal block of the function (5.6) can be obtained in closed form for any value of the 'anomaly' number b, with the result given in (5.7) above.

These operators will then have the following OPEs

$$\xi(z)\xi^\dagger(0) = \frac{1}{8}T(z)T(0) + \frac{b}{2z^4} + \frac{t(0)+T(0)\log(z)}{z^2} + \ldots, \tag{5.8}$$

and

$$t(z)\xi(0) = \frac{1}{4}T(z)\xi(0) - T(z)\xi(0)\log(z) + \frac{\xi'(0)}{2z} + ..., \tag{5.9}$$

and similarly for $\xi^\dagger$.

Consider now the OPEs given by Eqs. (2.7), (2.21), (2.24), (5.8) and (5.9). They were derived using CFT techniques alone, and the only assumption was the existence of the OPE (2.21), and the fermionic nature of the operators ξ and $\xi^\dagger$. Nevertheless, it can easily be verified that these OPEs are in fact covariant under the application of a global SUSY transformation according to Fig. 2, acting only on the chiral operators appearing in them. (Interestingly, these OPEs would not have been covariant if $\alpha \neq \frac{1}{8}$.) Thus we arrive at a remarkable conclusion: if a partner of stress energy tensor t (which always exists at $c = 0$ to avoid the $c \to 0$ catastrophe) is logarithmic, these *chiral* OPEs are automatically supersymmetric. This would have to be true even when the field theory was constructed in terms of replicas as opposed to supersymmetry, as long as there are *two* weight-two primary operators $\xi, \xi^\dagger$ with nonvanishing two-point function which are *fermionic*. (We know from (5.4) that there exists at least one such weight-two operator.)

5.2. *Comments on an 'Extended Algebra'*

In Section 3 we used the logarithmic 'algebra' formed [w] by the coefficients of the mode expansion of T and t, which we called L_n and ℓ_m. Specifically, we derived the commutation relation $[L_n, \ell_m]$, given by (3.5). It was enough to consider only this commutation relation, besides the Virasoro algebra (3.6), to arrive at the results obtained in that Section.

It may be natural to ask if there could exist a suitable consistent (full) extension of the Virasoro algebra, involving the modes ℓ_m. For example, one could consider, in addition to Virasoro descendants $L_{-n}|A\rangle$, also 'extended Virasoro descendants' such as e.g. $\ell_{-n}|A\rangle$. One may also ask about a possible generalization of null states, degenerate with respect to a suitably extended Virasoro algebra, involving now also the modes ℓ_{+n}.

Unfortunately, proposals along these lines have been difficult to implement. In order to have a closed algebra, one would also need the commutation relations $[\ell_n, \ell_m]$, which would have to be obtained from the OPE (2.24). Similarly, since we now see that the full set of weight-two fields involves not only the fields T and t, but also additional fields such as e.g. the fermionic

[w] 'Algebra' appears here in quotes because the commutator $[\ell_n, \ell_m]$ required by closure was not discussed; for an elaboration, see the paragraphs below.

fields ξ and $\xi^\dagger$, it seems natural at this point to try to establish the anti-commutation relations $\{\xi_n, \xi_m^\dagger\}$ as well. One of the difficulties with these commutators is the nonholomorphic aspect of these fields. Furthermore, regarding the second commutator (but in view of (5.4) this is also relevant for the first), the operator ξ, when acting primary operators, may generate other primary operators with different conformal weights. For example, for the specific values of b when the Kač null-vector of ξ vanishes, its OPE with other primary fields can be read off from the Kač table, and it obviously generates primaries with weights not related by integers. For arbitrary values of b when the Kač table is not available, it is not even clear how to find such an OPE. In any case, it would be interesting if a suitable extension of the Virasoro algebra, based on the additional structures presented in this paper could be developed, in one way or the other. This, however, would certainly have to be reserved for future work.

6. Conclusions

In this paper we examined the structure of conformal field theories (CFTs) with central charge $c = 0$. We focussed entirely on the holomorphic sector, leaving the gluing of holomorphic and antiholomorphic sectors for future work. One of the main features distinguishing these CFTs from those with $c \neq 0$ is the appearance of a logarithmic 'partner of the stress tensor' which is in general not holomorphic. It has a holomorphic part of conformal weight two, which we denoted by $t(z)$ (Section 2). The latter leads to a novel anomaly number b, distinguishing different CFTs with central charge $c = 0$. The number b is a unique characteristic of any given $c = 0$ CFT. (Basically, b plays the role of the 'effective central charge' $(\partial c(n)/\partial n)_{|n=0}$ of replica theories [27, 30]; recall the footnote below (4.3).) Interestingly, we saw in Section 3 that in theories with 'logarithmic' $t(z)$ the number b controls the question of whether certain Kač-table null-vectors do indeed vanish identically or whether they represent nonvanishing states with zero norm (like the $c = 0$ stress tensor). We found that the null-vectors indeed vanish for primary operators with nonvanishing two-point function (i) in the first two rows of the Kač table when $b = +5/6$, or (ii) in the first column of the Kač table with nonzero weight when $b = -5/8$. Only those (chiral) four-point functions which contain a Kač-table operator with vanishing null-vector will satisfy the familiar [1] differential equations. These results were obtained by considering suitably defined Laurent modes ℓ_n of the logarithmic partner $t(z)$ of the stress tensor, and by considering their commutation relations (3.5) with the ordinary Virasoro generators L_n. (This represents a 'partial extension' of the Virasoro algebra.) We showed in Section 5 that on grounds of consistency there must exist, besides the stress tensor and its partner $t(z)$, additional fields whose holomorphic parts have weight two, and nonvanishing two-point functions, when $t(z)$ is logarithmic. Remarkably, the simple assumption of a fermionic pair of such additional fields implies that the full set of chiral OPEs between all these weight-two fields is automatically covariant under the action of a global $U(1|1)$ supersymmetry. No SUSY was required at the outset. (Then, also, the full identity conformal block of these fermionic weight-two fields can be computed exactly for any value of b, with the result given in (5.7).) Indeed, a $U(1|1)$ multiplet of 'stress tensors' transforming in the same indecomposable representation occurs in any CFT which is known to possess a global $U(1|1)$ SUSY, the actual stress tensor being the singlet (see Section 4). Our results show that, at least at the purely holomorphic level, such a global SUSY is already a hidden symmetry in any $c = 0$ CFT possessing a logarithmic partner of the stress tensor $t(z)$, given there are two fermionic weight-two fields $\xi, \xi^\dagger$ with nonvanishing two-point function. We close by saying that it is tempting to speculate about a possible extension of the Virasoro algebra by the Laurent modes ℓ_n of the logarithmic partner $t(z)$ of the stress tensor, and by the corresponding modes of the other members of the 'stress tensor multiplet' mentioned above. Future work will have to show if such an extension can be constructed (Section 5.2).

Appendix A: Uniqueness of the 'anomaly' number b

In this appendix we will show that the OPE (2.21), which expresses the action of (infinitesimal) conformal transformations on the operator t, does not allow for two *different* operators $t_1(z)$ and $t_2(z)$, characterized by two *different* values of their respective 'anomaly' numbers $b_1 \neq b_2$, to coexist in a given theory.

The proof is simple. If both, $t_1(z)$ and $t_2(z)$, were present in the same theory, then we would be able to construct the (holomorphic) two-point function $\langle t_1(z_1) t_2(z_2) \rangle$. As in any CFT, this function must satisfy the constraints of global conformal invariance (there are no others for a two-point function). We will show that these constraints (ordinary differential equations) do not possess a solution, unless $b_1 = b_2$.

We start [x] by recalling that the OPE (2.21) yields the change of the operators $t_i(z)$, $(i = 1, 2)$ under an infinitesimal conformal transformation

[x] We note in passing that the transformation law under a finite conformal transformation $w = w(z)$ is $t(w) = (\frac{dz}{dw})^2 t(z) + [\ln(\frac{dz}{dw})]\ T(z)$.

$w(z) = z + \epsilon(z)$,

$$\delta_{\epsilon(z)}\ t_i(z) = \int_{C(z)} \frac{d\zeta}{2\pi i}\ \epsilon(\zeta)\ T(\zeta) t_i(z)$$
$$= \left(\frac{d}{dz}\epsilon(z)\right)[2t(z) + T(z)] + \epsilon(z)\frac{d}{dz}t(z) + \frac{b_i}{3!}\frac{d^3\epsilon(z)}{dz^3}\ , \qquad \text{(A.1)}$$

where $i = 1,\ 2$. The action of the global conformal group ($\mathrm{Sl}(2;\mathbf{C})$) corresponds to functions $\epsilon(z)$ which are 2nd order polynomials in z. Now consider the two-point function

$$\langle t_1(z_1) t_2(z_2)\rangle$$

which is a function only of $z_{12} = z_1 - z_2$, due to translational invariance (corresponding to $\epsilon(z) = constant$). Choosing $\epsilon(z) = \epsilon \cdot z$ (scale invariance) in (A.1) yields

$$\delta_{\epsilon(z)}\langle t_1(z_1)t_2(z_2)\rangle = \langle\left(\delta_{\epsilon(z)}t_1(z_1)\right)t_2(z_2)\rangle + \langle t_1(z_1)\left(\delta_{\epsilon(z)}t_2(z_2)\right)\rangle$$
$$= \left(\left[z_1\frac{d}{dz_1} + z_2\frac{d}{dz_2}\right] + 4\right)\langle t_1(z_1)t_2(z_2)\rangle + \frac{b_1+b_2}{z_{12}^4} = 0\,, \qquad \text{(A.2)}$$

or (using translational invariance)

$$z_{12}\frac{d}{dz_{12}}\left[(z_{12})^4\langle t_1(z_1)t_2(z_2)\rangle\right] + (b_1+b_2) = 0 \qquad \text{(A.3)}$$

which has the solution

$$(z_{12})^4\langle t_1(z_1)t_2(z_2)\rangle = -(b_1+b_2)\ \ \ln(z_{12}) + \mathrm{const.} \qquad \text{(A.4)}$$

Next, choosing $\epsilon(z) = \epsilon \cdot z^2$ (special conformal transformations) in (A.1),

$$\left\langle\left\{2z_1\left[2t_1(z_1) + T(z_1)\right] + z_1^2\frac{d}{dz_1}t_1(z_1)\right\}t_2(z_2)\right\rangle$$
$$+\left\langle t_2(z_2)\left\{2z_2\left[2t_2(z_2) + T(z_2)\right] + z_2^2\frac{d}{dz_2}t_2(z_2)\right\}\right\rangle = 0\,. \qquad \text{(A.5)}$$

Setting $z_2 = 0$ yields

$$z_1\frac{d}{dz_1}\left[(z_1)^4\langle t_1(z_1)t_2(z_2)\rangle\right] + 2\left[(z_1)^4\langle T(z_1)t_2(z_2)\rangle\right] = 0\,. \qquad \text{(A.6)}$$

This gives, using (2.21) and (A.4)

$$-(b_1+b_2) + 2b_2 = 0, \quad \text{or} \quad b_1 = b_2\,. \qquad \text{(A.7)}$$

This completes the proof.

Finally, we can read off from (A.4) the two-point function of the single operator $t = t_1 = t_2$,

$$\langle t(z_1)t(z_2)\rangle = \frac{-2b\ln(z_{12}) + \text{const}}{z_{12}^4}\,, \tag{A.8}$$

as in (2.22) of the main text.

We close Appendix A by recalling that (A.1) leads to the action of L_n in the usual way [1], by letting $\epsilon(z) \propto z^{n+1}$ $(n = -1, 0, 1, 2, ...)$. This yields, in particular, the relations in (B.10) below.

Appendix B: Computation of OPEs of Virasoro descendants

This appendix is devoted to the computation of OPEs such as for example (2.24). More generally, consider instead of the two operators $t(z)$ and $t(0)$ in that equation, two operators $O_1(z)$ and $O_2(0)$. We are interested in the descendants of some third operator of conformal weight h', appearing in the OPE $O_1(z)O_2(0)$.

Here we consider the OPE of the two not necessarily primary operators $O_1(z)$ and $O_2(z)$ of conformal weights h_1 and h_2,

$$O_1(z)O_2(0) = ... + \frac{1}{z^{h_1+h_2-h'}}\,X(z) + ...\,, \tag{B.1}$$

where the ellipsis denotes contributions from other primary operators. (Examples are the OPEs in (2.3), (2.16), (B.12), (2.21), (2.24).) In general, $X(z)$, which denotes the contributions to this OPE from a primary operator of conformal weight h', has the form

$$X(z) = X^{(0)}(z) + [\ln(z)]X^{(1)}(z) + [\ln^2(z)]X^{(2)}(z) + ...\,, \tag{B.2}$$

where $X^{(i)}(z)$, $(i = 0, 1, 2, ...)$ are power series in z whose coefficients are operators evaluated at the point $z = 0$, denoting descendants (as well as their logarithmic 'partners'). Explicitly, we have

$$X^{(i)}(z) = \sum_{n=0}^{\infty} z^n\ \hat{X}_n^{(i)}(0), \qquad (i = 0, 1, 2, ...)\,, \tag{B.3}$$

where $\hat{X}_n^{(i)}(0)$ is an operator of conformal weight raised by $+n$ as compared to $\hat{X}_0^{(i)}(0)$, or a (certain, to-be-determined) linear combination of such operators.

Below, we will be interested in the logarithmic derivative of $X(z)$. Expanding $X(z)$ as in (B.2) we have

$$\left(z\frac{d}{dz}\right)X(z)=\left\{\left(z\frac{d}{dz}\right)X^{(0)}(z)+X^{(1)}(z)\right\}+\ln(z)\left\{\left(z\frac{d}{dz}\right)X^{(1)}(z)+2X^{(2)}(z)\right\}$$
$$+\ln^2(z)\left\{\left(z\frac{d}{dz}\right)X^{(2)}(z)+3X^{(3)}(z)\right\}+\ldots. \tag{B.4}$$

Moreover, using the expansion (B.2), (B.3) one obtains

$$\frac{1}{z^n}\left[L_{+n},\, X^{(i)}\right]=\sum_{m=n}^{\infty}z^{m-n}\left[L_{+n},\, \hat{X}^{(i)}_m(0)\right]=\sum_{m=0}^{\infty}z^m\left[L_{+n},\, \hat{X}^{(i)}_{m+n}(0)\right]. \tag{B.5}$$

Note that $\left[L_{+n},\, \hat{X}^{(i)}_m(0)\right]$ has conformal weight $(m-n)$ and that the expression vanishes when $n > m$.

Our aim in this appendix is to establish a recursion relation to determine all the coefficients of the entire power series, starting from the first few (with lowest powers of z). The fact that this is possible means that the entire OPE is uniquely determined by its first few terms.

We start by considering the commutator of both sides of (B.1) with the Virasoro (raising) operator L_{+n}, $n \geq 0$ (it is actually enough to consider only $n = 0, 1, 2$ because the others are generated using the Virasoro algebra),

$$[L_{+n},\, O_1(z)O_2(0)] = [L_{+n},\, X(z)] + \text{other prim.} \tag{B.6}$$

The left-hand side can we written as

$$[L_{+n},\, O_1(z)O_2(0)] = ([L_{+n},\, O_1(z)])\, O_2(0) + O_1(z)\, ([L_{+n},\, O_2(0)])\,. \tag{B.7}$$

These commutators are given by simple expressions.

Examples of needed commutators

We give three examples of commutators which we will need:

(i) for $A_h(z)$ a Virasoro primary of conformal weight h

$$n=0,1,2,\ldots: \qquad [L_{+n},\, A_h(z)] = z^n\left(z\frac{d}{dz}+h(n+1)\right)A_h(z),$$
$$[L_{+n},\, A_h(0)] = h\,\delta_{n,0}\,A_h(0); \tag{B.8}$$

(ii) for the stress tensor

$$n = 0, 1, 2, ... : \qquad [L_{+n},\, T(z)] = z^n \left(z\frac{d}{dz} + 2(n+1) \right) T(z) + \frac{c}{12} n(n^2-1) z^{n-2},$$
$$[L_{+n},\ T(0)] = 2\delta_{n,0}\, T(0) + \delta_{n,2}\ \frac{c}{2}\ ; \tag{B.9}$$

(iii) for the 'partner' $t(z)$ of the stress tensor

$$n = 0, 1, 2, ... : \quad [L_{+n},\, t(z)] = z^n \left\{ \left(z\frac{d}{dz} + 2(n+1) \right) t(z) + (n+1) T(z) \right\} + \frac{b}{6} n(n^2-1) z^{n-2}\,,$$
$$[L_{+n},\, t(0)] = \delta_{n,0} \left(2t(0) + T(0) \right) + \delta_{n,2}\ b\,. \tag{B.10}$$

B.1. *The OPE of two primary operators, $A_{h_1}(z)A_{h_2}(0)$*

We now proceed to derive the recursion relations for the OPE of two primary operators of conformal weights h_1 and h_2, respectively,

$$O_1 = A_{h_1}, \qquad O_1 = A_{h_2}\,. \tag{B.11}$$

We are interested in the descendants of a third primary operator $A_{h'}$ of conformal weight h' appearing in this OPE, which we characterize by the (operator-valued) function $X(z)$,

$$A_{h_1}(z) A_{h_2}(0) = ... + \frac{1}{z^{h_1+h_2-h'}} X(z) + ...$$
$$= ... + \frac{1}{z^{h_1+h_2-h'}} \left\{ A_{h'}(0) + a_1\ z\ L_{-1} A_{h'}(0) + O(z^2) \right\} + ...\,, \tag{B.12}$$

where the ellipsis denote contributions from other primary operators. Making use of (B.6), (B.7), (B.8) for $n \geq 1$ yields

$$z^n \left[\left(z\frac{d}{dz} \right) + h_1(n+1) \right] A_{h_1}(z) A_{h_2}(0) = ... + \frac{1}{z^{h_1+h_2-h'}} \left[L_{+n}, X(z) \right] + ... \tag{B.13}$$

which becomes [y]

$$n \geq 1: \quad \left(z\frac{d}{dz} + nh_1 - h_2 + h'\right) X(z) = \frac{1}{z^n}\left[L_{+n},\, X(z)\right]. \tag{B.14}$$

Using (B.4) and (B.5) in (B.14) we obtain a recursion for the coefficients

$$\begin{aligned} n \geq 1: \quad [m + nh_1 - h_2 + h']\hat{X}_m^{(0)} + \hat{X}_m^{(1)} &= \left[L_{+n},\, \hat{X}_{m+n}^{(0)}(0)\right], \\ [m + nh_1 - h_2 + h']\hat{X}_m^{(1)} + 2\hat{X}_m^{(2)} &= \left[L_{+n},\, \hat{X}_{m+n}^{(1)}(0)\right], \\ [m + nh_1 - h_2 + h']\hat{X}_m^{(2)} + 3\hat{X}_m^{(3)} &= \left[L_{+n},\, \hat{X}_{m+n}^{(2)}(0)\right], \end{aligned} \tag{B.15}$$

etc.. Upon choosing $n = 1, 2$, these equations determine the coefficients $\hat{X}_m^{(0)}(0), \hat{X}_m^{(1)}(0), \hat{X}_m^{(2)}(0)$ for higher values of the index from those with a lower index.

For $n = 0$ (dilations) we have

$$[L_0, A_{h_1}(z)A_{h_2}(0)] = \left(\left(z\frac{d}{dz}\right) + (h_1 + h_2)\right) A_{h_1}(z)A_{h_2}(0), \tag{B.16}$$

or

$$\left(\left(z\frac{d}{dz}\right) + h'\right) X(z) = [L_0, X(z)]. \tag{B.17}$$

This yields for the coefficients of $X(z)$

$$\begin{aligned} (m + h')\hat{X}_m^{(0)} + \hat{X}_m^{(1)} &= \left[L_0,\, \hat{X}_m^{(0)}\right], \\ (m + h')\hat{X}_m^{(1)} + 2\hat{X}_m^{(2)} &= \left[L_0,\, \hat{X}_m^{(1)}\right], \\ (m + h')\hat{X}_m^{(2)} + 3\hat{X}_m^{(3)} &= \left[L_0,\, \hat{X}_m^{(2)}\right], \end{aligned} \tag{B.18}$$

etc..

B.2. *The OPE $T(z)T(0) \sim 1$ at $c = 0$*

In order to compute this OPE we set

$$O_1 = O_2 = T \tag{B.19}$$

[y] We used $z^x\left(z\frac{d}{dz}\right)\left(z^{-x}X(z)\right) = \left(z\frac{d}{dz} - x\right)X(z)$.

and $h' = 0$ in (B.1). Making use of (B.6), (B.7), (B.9) for $n \geq 1$ we obtain

$$n \geq 1 : \qquad z^n \left\{ \left(z \frac{d}{dz} + 2(n+1) \right) T(z)\; T(0) \right\} = \frac{1}{z^4} \left[L_{+n},\, X \right]. \tag{B.20}$$

This has the same form as the equation (B.14) obtained above, for two conformal weight $= 2$ primary operators A_2 (i.e. $h_1 = h_2 = 2$). This is of course expected, because the stress tensor $T(z)$ is weight-two primary, at $c = 0$. These two types of primary weight-two operators differ only by the fact that one (the stress tensor) has a vanishing, and the other (A_2) a nonvanishing two-point function. The recursion relations are thus identical to (B.15), with $h_1 = h_2 = 2$. But the terms in $X(z)$ with small powers of z, i.e. the initial conditions of the recursion relations, are different in the two cases. Since for the stress tensor $T(z)$ the initial conditions of the recursion do not contain any logarithms, this continues to be case for all terms, and the recursion relations (B.15) simplify further,

$$n \geq 1 : \qquad [m + 2(n-1)] \hat{X}^{(0)}_m = \left[L_{+n},\, \hat{X}^{(0)}_{m+n}(0) \right]. \tag{B.21}$$

B.3. *The OPE* $t(z)A_h(0) \sim A_h(0)$

In this section of Appendix B we determine the terms appearing in the OPE (3.2) of Subsection 3.1. We write this OPE in the form

$$t(z)A_h(0) := \frac{1}{z^2}\, X(z) + \Re(z)\,, \tag{B.22}$$

where $\Re(z)$ denotes all those terms in this OPE which contain noninteger powers of z.

(i) Applying (B.6), (B.7) to (B.22) yields

$$\begin{aligned} &n \geq 1 : \\ &z^n \left\{ \left(z \frac{d}{dz} + 2(n+1) \right) t(z) + (n+1) T(z) + \frac{b}{6}\, n(m^2-1) z^{-2} \right\} A_h(0) \\ &= \left[L_n, \frac{1}{z^2}\, X(z) + \Re(z) \right]. \end{aligned} \tag{B.23}$$

This becomes, abbreviating

$$T(z)A_h(0) := \frac{1}{z^2}\, Y(z), \tag{B.24}$$

$$n \geq 1 :$$
$$z^2 \left(z\frac{d}{dz} + 2(n+1) \right) z^{-2} X(z) + (n+1)Y(z) + \frac{b}{6} n(n^2-1)A(0)$$
$$= \frac{1}{z^n} \left[L_n, X(z) \right] . \tag{B.25}$$

Using the footnote in Subsection B.1 the above reduces to

$$n \geq 1 :$$
$$\left(z\frac{d}{dz} + 2n \right) X(z) + (n+1)Y(z) + \frac{b}{6}\, n(n^2-1)A(0)$$
$$= \frac{1}{z^n} \left[L_n, X(z) \right] . \tag{B.26}$$

(ii) Similarly, for $n = 0$ we obtain from (B.6), (B.7) and (B.22)

$$n = 0 :$$
$$\left(z\frac{d}{dz} + 2 + h \right) t(z)A_h(0) + T(z)A_h(0) = \left[L_0, \frac{1}{z^2} X(z) + \Re(z) \right] , \tag{B.27}$$

leading to

$$n = 0 : \qquad \left(z\frac{d}{dz} + h \right) X(z) + Y(z) = \left[L_0, X(z) \right] . \tag{B.28}$$

(iii) Finally, inserting the decomposition (B.2), using (B.4) as well as (B.5), and recalling that the quantity $Y(z)$ defined in (B.24) has no terms proportional to $\ln(z)$, we obtain

∗ from (B.26),

$$n \geq 1 : \qquad \left\{ [m+2n]\hat{X}^{(0)}_m + (n+1)\hat{Y}^{(0)}_m + \frac{b}{6}\, n(n^2-1)A(0) + \hat{X}^{(1)}_m \right\}$$
$$= \left[L_{+n},\ \hat{X}^{(0)}_{m+n}(0) \right] ,$$
$$\left\{ [m+2n]\hat{X}^{(1)}_m + 2\hat{X}^{(2)}_m \right\} = \left[L_{+n},\ \hat{X}^{(1)}_{m+n}(0) \right] , \tag{B.29}$$

etc.

∗ from (B.18),

$$n = 0 : \qquad \left\{ (m+h)\hat{X}^{(0)}_m + \hat{Y}^{(0)}_m + \hat{X}^{(1)}_m \right\} = \left[L_0,\ \hat{X}^{(0)}_m \right] ,$$
$$\left\{ (m+h)\hat{X}^{(1)}_m + 2\hat{X}^{(2)}_m \right\} = \left[L_0,\ \hat{X}^{(1)}_m \right] ,$$

etc..

B.4. *The OPE $t(z)t(0) \sim 1$, $(c=0)$*

In order to compute this OPE we set

$$O_1 = O_2 = t \tag{B.30}$$

and $h' = 0$ in (B.1). Making use of (B.6), (B.7), (B.10) for $n \geq 1$

$$\left\{ \left[(z\frac{d}{dz} + 2(n+1) \right] t(z) + (n+1)T(z) + \frac{b}{6}\, n(n^2-1)z^{-2} \right\} t(0)$$
$$+ t(z) b\delta_{n,2} z^{-2} = \frac{1}{z^4}\ \frac{1}{z^n}[L_{+n}, X] + ..., \quad (n \geq 1). \tag{B.31}$$

We now use the definition

$$Y(z) := \{ b + z^2 \left(2t(0) + T(0) \right) + z^3 L_{-1} t(0) + ... \} \tag{B.32}$$

for the term arising from the OPE $T(z)t(0) = Y(z)/z^4$, discussed in (2.21). Note that the so-defined function $Y(z)$ has an analytic z-dependence, and hence no logarithms. We denote this fact by writing (in view of the notation used in (B.2))

$$Y(z) = Y^{(0)}(z)\,. \tag{B.33}$$

Thus, we may write this as

$$\left[(z\frac{d}{dz} + 2(n-1) \right] X(z) + (n+1) Y^{(0)}(z)$$
$$+ z^2 \left[\frac{b}{6}\, n(n^2-1) t(0) + b\delta_{n,2} t(z) \right] = \frac{1}{z^n}[L_{+n}, X(z)], \tag{B.34}$$

at $n \geq 1$. To simplify the notation, we also define

$$Z(n;z) := z^2 \left[\frac{b}{6} n(n^2-1)t(0) + b\delta_{n,2} t(z) \right], \tag{B.35}$$

and

$$Z(0;z) = Z(1;z) = 0, \tag{B.36}$$

Moreover,

$$Z(2;z) = z^2 b \left[t(0) + t(z) \right], \quad Z(n;z) := z^2\, \frac{b}{6}\, n(n^2-1) t(0), \tag{B.37}$$

for $n \geq 3$, where

$$t(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Big(\frac{d}{dz}\Big)^n t(0), \tag{B.38}$$

in parallel to

$$\begin{aligned} &T(z) = T(z)\mathbf{1}(0) = \sum_{n=0}^{\infty} \; z^n L_{-2-n}\mathbf{1}(0) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Big(\frac{d}{dz}\Big)^n T(0), \\ &T(0) = L_{-2}\mathbf{1}(0). \end{aligned} \tag{B.39}$$

Hence, we see that

$$Z(n; z) = Z^{(0)}(n; z) \tag{B.40}$$

contains no logarithms.

Now, we may write the previous equation, (B.34), in the following final form (recalling the definitions (B.32) and (B.35))

$$\left[\Big(z\frac{d}{dz}\Big) + 2(n-1)\right] X(z) + (n+1)Y^{(0)}(z) + Z^{(0)}(z) = \frac{1}{z^n}\,[L_{+n}, X], \quad (n \geq 1).$$

The recursion for the coefficients now reads

$$\begin{aligned} &n \geq 1: \\ &[m + 2(n-1)]\hat{X}_m^{(0)} + \hat{X}_m^{(1)} + (n+1)\hat{Y}_m^{(0)} + \hat{Z}_m^{(0)}(n) = \Big[L_{+n},\ \hat{X}_{m+n}^{(0)}(0)\Big], \\ &[m + 2(n-1)]\hat{X}_m^{(1)} + 2\hat{X}_m^{(2)} = \Big[L_{+n},\ \hat{X}_{m+n}^{(1)}(0)\Big], \\ &[m + 2(n-1)]\hat{X}_m^{(2)} = \Big[L_{+n},\ \hat{X}_{m+n}^{(2)}(0)\Big]. \end{aligned} \tag{B.41}$$

(We have omitted terms containing a triple power of the logarithm, as they will not be generated.)

For $n = 0$ we obtain from (B.6), (B.7), (B.10)

$$[L_0, t(z)t(0)] = \left\{\left(\Big(z\frac{d}{dz}\Big) + 2\right) t(z) + T(z)\right\} t(0) + t(z)\big(2t(0) + T(0)\big) \tag{B.42}$$

or

$$\Big(z\frac{d}{dz}\Big) X(z) + Y^{(0)}(z) + z^4 t(z)T(z) = [L_0, X(z)]\,, \tag{B.43}$$

leading to

$$m\hat{X}^{(0)}_m + \hat{X}^{(1)}_m + \hat{Y}^{(0)}_m + z^4 t(z) T(z) = \left[L_0,\, \hat{X}^{(0)}_m(0)\right],$$
$$m\hat{X}^{(1)}_m + 2\hat{X}^{(2)}_m = \left[L_0,\, \hat{X}^{(1)}_m(0)\right],$$
$$m\hat{X}^{(2)}_m = \left[L_0,\, \hat{X}^{(2)}_m(0)\right]. \qquad \text{(B.44)}$$

B.5. *Summary of Recursions*

<u>*Notation:*</u>

$$X(z) = X^{(0)}(z) + [\ln(z)]X^{(1)}(z) + [\ln^2(z)]X^{(2)}(z) + \ldots.$$
$$X^{(i)}(z) = \sum_{n=0}^{\infty} z^n\ \hat{X}^{(i)}_n(0), \qquad (i = 0, 1, 2, \ldots). \qquad \text{(B.45)}$$

(We omit the position (0) in the formulas below.)

- $A_h(z)A_h(0) = \frac{1}{z^{2h}} X^{(A)}(z) + \ldots, \quad \big(\text{identity operator,}\ \ h' = 0\big)$, (B.46)

$$m = 0, 1, 2, \ldots$$
$$n \geq 1: \quad [m + h(n-1)]\hat{X}^{(A;0)}_m + \hat{X}^{(A;1)}_m = \left[L_{+n},\, \hat{X}^{(A;0)}_{m+n}\right],$$
$$[m + h(n-1)]\hat{X}^{(A;1)}_m = \left[L_{+n},\, \hat{X}^{(A;1)}_{m+n}\right], \qquad \text{(B.47)}$$

$$n = 0: \quad m\hat{X}^{(0)}_m + \hat{X}^{(1)}_m = \left[L_0,\, \hat{X}^{(0)}_m\right],$$
$$m\hat{X}^{(1)}_m = \left[L_0,\, \hat{X}^{(1)}_m\right]. \qquad \text{(B.48)}$$

- $T(z)T(0) = \frac{1}{z^4}\, X^{(T)}(z)$,

$$n \geq 1, \quad m = 0, 1, 2, \ldots:$$
$$[m + 2(n-1)]\hat{X}^{(T;0)}_m = \left[L_{+n},\ \hat{X}^{(T;0)}_{m+n}\right]. \qquad \text{(B.49)}$$

- $t(z)A_h(0) = \frac{1}{z^2}\, X^{(tA)}(z) + \Re(z), \quad T(z)A_h(0) = \frac{1}{z^2} Y(z),$

$$\begin{aligned}
&m = 0, 1, 2, ...\\
&n \geq 1: \quad [m+2n]\hat{X}_m^{(tA;0)} + (n+1)\hat{Y}_m^{(0)}\\
&\qquad + \frac{b}{6}\, n(n^2-1)A(0) + \hat{X}_m^{(tA;1)} = \left[L_{+n},\ \hat{X}_{m+n}^{(tA;0)}\right],\\
&\qquad [m+2n]\hat{X}_m^{(tA;1)} + 2\hat{X}_m^{(tA;2)} = \left[L_{+n},\ \hat{X}_{m+n}^{(tA;1)}\right],
\end{aligned} \tag{B.50}$$

etc.

$$\begin{aligned}
n = 0: \qquad &(m+h)\hat{X}_m^{(tA;0)} + \hat{Y}_m^{(0)} + \hat{X}_m^{(tA;1)} = \left[L_0,\ \hat{X}_m^{(tA;0)}\right],\\
&(m+h)\hat{X}_m^{(tA;1)} + 2\hat{X}_m^{(tA;2)} = \left[L_0,\ \hat{X}_m^{(tA;1)}\right],
\end{aligned} \tag{B.51}$$

etc. (Note that the combination $\hat{Y}_m^{(0)} + \hat{X}_m^{(tA;1)}$ appearing in the first of these two equations must vanish, because $\hat{X}_m^{(tA;0)}$ has weight $(h+m)$. This determines $\hat{X}_m^{(tA;1)}$ with the same result as in Section B.7 below.)

- $t(z)t(0) = \dfrac{1}{z^4} X^{(t)}(z) + \ldots, \quad T(z)t(0) = \dfrac{1}{z^4} Y^{(t)}(z),$

$$\begin{aligned}
&n \geq 1;\ m = 0, 1, 2, ... :\\
&[m + 2(n-1)]\hat{X}_m^{(t;0)} + \hat{X}_m^{(t;1)} + (n+1)\hat{Y}_m^{(t;0)} + \hat{Z}_m^{(t;0)}(n) = \left[L_{+n},\ \hat{X}_{m+n}^{(t;0)}\right],\\
&[m + 2(n-1)]\hat{X}_m^{(t;1)} + 2\hat{X}_m^{(t;2)} = \left[L_{+n},\ \hat{X}_{m+n}^{(t;1)}\right],\\
&[m + 2(n-1)]\hat{X}_m^{(t;2)} = \left[L_{+n},\ \hat{X}_{m+n}^{(t;2)}\right],
\end{aligned} \tag{B.52}$$

$$\begin{aligned}
n = 0: \quad &m\hat{X}_m^{(t;0)} + \hat{X}_m^{(t;1)} + \hat{Y}_m^{(0)} + z^4 t(z)T(z) = \left[L_0,\ \hat{X}_m^{(t;0)}\right],\\
&m\hat{X}_m^{(t;1)} + 2\hat{X}_m^{(t;2)} = \left[L_0,\ \hat{X}_m^{(t;1)}\right],\\
&\left\{m\hat{X}_m^{(t;2)}\right\} = \left[L_0,\ \hat{X}_m^{(t;2)}\right].
\end{aligned} \tag{B.53}$$

(Again, we have omitted terms containing a triple power of the logarithm, as they will not be generated.)

B.6. *Details of OPE $A_h(z)A_h(z) \sim 1$*

The form of the descendants of the identity operator appearing in the OPE $A_h(z)A_h(z)$ of two primary operators, in the normalization of (2.16), corresponds to the coefficients

$$X^{(A;0)} = 1 + z^2 \frac{h}{b} t(0) + O(z^3),$$
$$X^{(A;1)} = \qquad z^2 \frac{h}{b} T(0) + O(z^3)\,. \qquad \text{(B.54)}$$

Let us illustrate how the order $O(z^2)$ terms, and similarly all the others, are obtained from the leading term in (B.54) by applying the recursion.
The lowest order terms of (B.54) read

$$X_0^{(A;0)} = 1; \qquad X_0^{(A;1)} = 0\,. \qquad \text{(B.55)}$$

Using (B.47) with $m = 0$ and $n = 2$ leads to

$$h + 0 = \left[L_{+2}, X_2^{(A;0)}\right]\,, \qquad \text{(B.56)}$$

$$0 = \left[L_{+2}, X_2^{(A;1)}\right]\,, \qquad \text{(B.57)}$$

which implies, using (B.9), (B.10)

$$X_2^{(A;0)} = \frac{h}{b}\left(t(0) + \alpha\, T(0)\right),$$
$$X_2^{(A;1)} = \frac{h}{b}\,\beta\, T(0)\,. \qquad \text{(B.58)}$$

Using (B.48) with $m = 2$ we get

$$2X_2^{(A;0)} + X_2^{(A;1)} = \left[L_{+0}, X_2^{(A;0)}\right]\,,$$
$$2X_2^{(A;1)} = \left[L_{+0}, X_2^{(A;1)}\right]\,, \qquad \text{(B.59)}$$

which yields, when setting the arbitrary constant $\alpha \to 0$

$$2t(0) + \beta\; T(0) = 2t(0) + T(0)\,,$$
$$2\beta\; T(0) = 2\beta\, T(0)\,. \qquad \text{(B.60)}$$

Hence we have found $\beta = 1$ ($\alpha = 0$), in agreement with (B.54). (Note that α is arbitrary because if corresponds (at $c = 0$) to the contribution of a primary operator $T(z)$ (the stress tensor) to the OPE $A_h(z)A_h(z)$.)

B.7. *Details of OPE* $t(z)A_h(z) \sim A_h(z)$

We begin by writing down the leading terms in the OPEs (B.2), (B.22), (B.24),

$$\begin{aligned} X^{(tA;0)} &= 0 \qquad\quad + O(z)\,, \\ X^{(tA;1)} &= -hA(0) + O(z)\,, \\ Y(z) &= hA(0) \quad + zL_{-1}A(0) + O(z^2)\,. \end{aligned} \tag{B.61}$$

Recall that in the notation of (3.2)

$$X^{(tA;0)} = z\ \ell_{-1}A(0) + z^2\ \ell_{-2}A(0) + \ldots\,. \tag{B.62}$$

a) We start with (B.50) for $m = 0$ and $n = 1$, which reads

$$\begin{aligned} 2\hat{X}_0^{(tA;0)} + 2\hat{Y}_0^{(0)} + \hat{X}_0^{(tA;1)} &= \left[L_{+1},\ \hat{X}_1^{(tA;0)}(0)\right], \\ 2\hat{X}_0^{(tA;1)} + 2\hat{X}_0^{(tA;2)} &= \left[L_{+1},\ \hat{X}_1^{(tA;1)}(0)\right]. \end{aligned} \tag{B.63}$$

Using the information contained in the lowest order terms of (B.61) this becomes

$$\begin{aligned} hA(0) &= \left[L_{+1},\ \hat{X}_1^{(tA;0)}(0)\right], \\ -2hA(0) &= \left[L_{+1},\ \hat{X}_1^{(tA;1)}(0)\right]. \end{aligned} \tag{B.64}$$

The solutions of these equations are

$$\begin{aligned} \hat{X}_1^{(tA;0)}(0) = -\frac{1}{2}\hat{X}_1^{(tA;1)}(0) &= \frac{1}{2}L_{-1}A(0) + \gamma_1\ \tilde{A}_{h+1}(0)\,, \\ \hat{X}_1^{(tA;1)}(0) \qquad\qquad\qquad &= -L_{-1}A(0) - \delta_1\ \tilde{B}_{h+1}(0)\,, \end{aligned} \tag{B.65}$$

where γ_1, δ_1 are so-far arbitrary coefficients, and $\tilde{A}_{h+1}(0)$, $\tilde{B}_{h+1}(0)$, could be any primary operators of conformal weight $(h+1)$ (if those exist). Making use of (B.51) with m=1 shows however that $\delta_1 = 0$, whereas γ_1 remains arbitrary. $\tilde{A}_{h+1}(0)$ represents the null-vector mentioned above Eq. (3.9); using the notation (B.62),

$$\ell_{-1}A(0) - \frac{1}{2}L_{-1}A(0) = \gamma_1\ \tilde{A}_{h+1}(0)\,. \tag{B.66}$$

If the particular theory under consideration does not have a weight-$(h+1)$ primary operator, then the extra primary in the first equation of (B.65) is also absent.

b) We continue with (B.50) for $m = 0$ and $n = 2$, which reads

$$4\hat{X}_0^{(tA;0)} + 3\hat{Y}_0^{(0)} + bA(0) + \hat{X}_0^{(tA;1)} = \left[L_{+2},\, \hat{X}_2^{(tA;0)}(0)\right],$$
$$4\hat{X}_0^{(tA;1)} + 2\hat{X}_0^{(tA;2)} = \left[L_{+2},\, \hat{X}_2^{(tA;1)}(0)\right]. \tag{B.67}$$

Using again the information contained in the lowest-order terms of (B.61) we arrive at

$$(2h+b)A(0) = \left[L_{+2},\, \hat{X}_2^{(tA;0)}(0)\right],$$
$$-4hA(0) = \left[L_{+2},\, \hat{X}_2^{(tA;1)}(0)\right]. \tag{B.68}$$

c) Furthermore, continuing with (B.50) for $m = 1$ and $n = 1$, we get

$$3\hat{X}_1^{(tA;0)} + 2\hat{Y}_1^{(0)} + \hat{X}_1^{(tA;1)} = \left[L_{+1},\, \hat{X}_2^{(tA;0)}(0)\right],$$
$$3\hat{X}_1^{(tA;1)} + 2\hat{X}_1^{(tA;2)} = \left[L_{+1},\, \hat{X}_2^{(tA;1)}(0)\right]. \tag{B.69}$$

Using the solution (B.65), this becomes (upon setting $\alpha = 0$)

$$3\left(\frac{1}{2}\right) L_{-1}A(0) + L_{-1}A(0) = \frac{5}{2}\, L_{-1}A(0) = \left[L_{+1},\, \hat{X}_2^{(tA;0)}(0)\right],$$
$$-3L_{-1}A(0) = \left[L_{+1},\, \hat{X}_2^{(tA;1)}(0)\right]. \tag{B.70}$$

Let us summarize parts b) and c): the equations (B.68) and (B.70),

$$(2h+b)A(0) = \left[L_{+2},\, \hat{X}_2^{(tA;0)}(0)\right],$$
$$-4hA(0) = \left[L_{+2},\, \hat{X}_2^{(tA;1)}(0)\right],$$
$$(5/2)L_{-1}A(0) = \left[L_{+1},\, \hat{X}_2^{(tA;0)}(0)\right],$$
$$-3L_{-1}A(0) = \left[L_{+1},\, \hat{X}_2^{(tA;1)}(0)\right], \tag{B.71}$$

represent four equations for the four unknowns $\alpha^{(0)}, \beta^{(0)}$ and $\alpha^{(1)}, \beta^{(1)}$, which determine the coefficients $\hat{X}_2^{(tA;0)}(0)$ and $\hat{X}_2^{(tA;1)}(0)$, respectively, through

$$\hat{X}_2^{(tA;0)}(0) = \left(\alpha^{(0)} L_{-2} + \beta^{(0)} (L_{-1})^2\right) A(0) + \gamma_2\ \tilde{A}_{h+2}(0)\,,$$
$$\hat{X}_2^{(tA;1)}(0) = \left(\alpha^{(1)} L_{-2} + \beta^{(1)} (L_{-1})^2\right) A(0)\,. \tag{B.72}$$

Here the coefficient γ_2 remains undetermined, and $\tilde{A}_{h+2}(0)$ is any primary of weight $(h+2)$. In the notation of (B.62)

$$\ell_{-2}A(0) - \left(\alpha^{(0)}L_{-2} + \beta^{(0)}(L_{-1})^2\right)A(0) = \gamma_2\ \tilde{A}_{h+2}(0)\,. \qquad \text{(B.73)}$$

(As before, a similar contribution to $\hat{X}_2^{(tA;1)}(0)$ vanishes by (B.51).)

B.8. *Details of the OPE $t(z)t(z) \sim 1$*

The form of the OPE $t(z)t(z)$ in (2.24) corresponds to the following coefficients

$$\begin{aligned}
X^{(t;0)} &= \qquad z^2 t(0) \qquad + \frac{1}{2}z^3 L_{-1}t(0) + O(z^4)\,,\\
X^{(t;1)} &= -2b + z^2[-4t(0) - T(0)] + \frac{1}{2}z^3 L_{-1}[-4t(0) - T(0)] + O(z^4)\,,\\
X^{(t;2)} &= \qquad z^2(-2)T(0) \qquad + \frac{1}{2}z^3(-2)L_{-1}T(0) + O(z^4)\,. \qquad \text{(B.74)}
\end{aligned}$$

Let us derive this OPE from the most singular term,

$$X_0^{(t;0)} = 0, \quad X_0^{(t;1)} = -2b, \quad X_0^{(t;2)} = 0\,, \qquad \text{(B.75)}$$

by applying the recursion (B.52).

To this end, we first make use of (B.9) and (B.10) with $n = 0, 1, 2$ to obtain

$$[L_0, T(0)] = 2T(0), \qquad [L_0, t(0)] = 2t(0) + T(0)\,, \qquad \text{(B.76)}$$

$$[L_{+1}, T(0)] = 0, \qquad [L_{+1}, t(0)] = 0\,, \qquad \text{(B.77)}$$

$$[L_{+2}, T(0)] = 0, \qquad [L_{+2}, t(0)] = b\,, \qquad \text{(B.78)}$$

$$[L_{+2}, L_{-1}T(0)] = 0, \qquad [L_{+2}, L_{-1}t(0)] = 0\,. \qquad \text{(B.79)}$$

The three equations (B.52) read for the special case $m = 0, n = 2$

$$\begin{aligned}
0 &= \left[L_{+1},\ \hat{X}_1^{(t;0)}\right], \qquad \{0 - 2b + 3b\} = \left[L_{+2},\ \hat{X}_2^{(t;0)}\right],\\
0 &= \left[L_{+1},\ \hat{X}_1^{(t;1)}\right], \qquad \{2(-2b) + 0\} = \left[L_{+2},\ \hat{X}_2^{(t;1)}\right],\\
0 &= \left[L_{+1},\ \hat{X}_1^{(t;2)}\right], \qquad 0 = \left[L_{+2},\ \hat{X}_2^{(t;2)}\right], \qquad \text{(B.80)}
\end{aligned}$$

which determines the right-hand side up to primary operators (i.e. $T(0)$ in this case, which we added below with undetermined coefficients α, β, γ),

$$\begin{aligned} \hat{X}_1^{(t;0)} &= 0\,, \qquad \hat{X}_2^{(t;0)} = t(0) + \alpha T(0)\,, \\ \hat{X}_1^{(t;1)} &= 0\,, \qquad \hat{X}_2^{(t;1)} = -4t(0) + \beta T(0)\,, \\ \hat{X}_1^{(t;2)} &= 0\,, \qquad \hat{X}_2^{(t;2)} = \gamma\, T(0)\,. \end{aligned} \tag{B.81}$$

The remaining terms are determined by using scale invariance, i.e. by (B.44) in the special case $m = 2$,

$$\begin{aligned} 2(t+\alpha T) + (-4t+\beta T) + 2(2t+T) &= \left[L_0,\ \hat{X}_2^{(0)}(0)\right] = 2t + T\,, \\ 2(-4t+\beta T) + 2\gamma T &= \left[L_0,\ \hat{X}_m^{(1)}(0)\right] = (-4)(2t+T) + 2\beta T\,, \\ 2\gamma T &= \left[L_0,\ \hat{X}_m^{(2)}(0)\right] = 2\gamma T\,, \end{aligned} \tag{B.82}$$

or

$$2\alpha + \beta = -1, \qquad \gamma = -2\,. \tag{B.83}$$

In the first of (B.81) we can always change α by redefinitions as in (2.23), and here we choose $\alpha = 0$. This yields $\beta = -1, \gamma = -2$ in agreement with (B.74).

B.9. *Subtraction of log and log-squared terms from $t(z)t(0)$ OPE*

In this subsection we consider the contribution of (the Virasoro representation, or 'conformal family' [1] of) the identity operator to the linear combination of OPEs discussed in (5.4) of Section (5.1),

$$t(z)t(0) + 4A_2(z)A_2(0)\ln(z) - T(z)T(0)\ln^2(z) + \frac{1}{2}T(z)T(0)\ln(z)\,. \tag{B.84}$$

We have set $\alpha = 0$ for convenience (but it can easily be reinstated). Our aim is to demonstrate that both single and double powers of $\ln(z)$ are absent from this expression to all orders in z.

- *Log-squared terms:*

It is clear that the term proportional to $\ln^2(z)$ cancels; the expansion coefficients for this linear combination are

$$\hat{X}_m^{2;total} := \hat{X}_m^{(t;2)} + 4\hat{X}_m^{(A;1)} - \hat{X}_m^{(T;0)}\,. \tag{B.85}$$

The recursion is the *same* for all three summands, so that we obtain the combined recursion

$$[m + 2(n-1)]\hat{X}_m^{2;total} = \left[L_{+n}, \hat{X}_{m+n}^{2;\,\mathrm{total}}\right]. \tag{B.86}$$

Now, since the expression $\hat{X}_m^{2;total}$ vanishes for small values of m, it in fact vanishes for all values of m as a consequence of this recursion relation.

- *Terms with a single power of log:*

  Next consider the terms proportional to a single power of $\ln(z)$; the expansion coefficients are

$$\hat{X}_m^{1;total} := \hat{X}_m^{(t;1)} + 4\hat{X}_m^{(A;0)} + \frac{1}{2}\hat{X}_m^{(T;0)}. \tag{B.87}$$

The relevant recursion relations are

$$\begin{aligned} [m + 2(n-1)]\hat{X}_m^{(t;1)} + 2\hat{X}_m^{(t;2)} &= \left[L_{+n},\ \hat{X}_{m+n}^{(t;1)}\right], \\ [m + 2(n-1)]4\hat{X}_m^{(A;0)} + 4\hat{X}_m^{(A;1)} &= \left[L_{+n},\ 4\hat{X}_{m+n}^{(A;0)}\right], \\ [m + 2(n-1)]\frac{1}{2}\hat{X}_m^{(T;0)} &= \left[L_{+n},\ \frac{1}{2}\hat{X}_{m+n}^{(T;0)}\right]. \end{aligned}$$

This can be combined into the *key equation*

$$[m + 2(n-1)]\hat{X}_m^{1;total} + 2\left\{\hat{X}_m^{(t;2)} + 2\hat{X}_m^{(A;1)}\right\} = \left[L_{+n},\ \hat{X}_{m+n}^{1;total}\right]. \tag{B.88}$$

Note that the expression

$$2\left\{\hat{X}_m^{(t;2)} + 2\hat{X}_m^{(A;1)}\right\} \tag{B.89}$$

is an inhomogeneity, which feeds into the recursion 'externally'.

We see from the third of (B.74) and the second of (B.54) that

$$2\left\{\hat{X}_m^{(t;2)} + 2\hat{X}_m^{(A;1)}\right\} = z^2\{-2T(0) + 2T(0)\} + O(z^3) = 0 + O(z^3). \tag{B.90}$$

Furthermore, from the second of (B.47) and the third of (B.52) we see that the quantity $\left\{\hat{X}_m^{(t;2)} + 2\hat{X}_m^{(A;1)}\right\}$ satisfies the recursion in the second of (B.47). Since the $m = 0, 1, 2$ coefficients of the recursion vanish, all coefficients vanish. Hence,

$$2\left\{\hat{X}_m^{(t;2)} + 2\hat{X}_m^{(A;1)}\right\} = 0. \tag{B.91}$$

In conclusion, the recursion in (B.88) now reduces to

$$[m+2(n-1)]\hat{X}_m^{1;\text{total}} = \left[L_{+n},\ \hat{X}_{m+n}^{1;\text{total}}\right]. \tag{B.92}$$

One finds by inspection that $\hat{X}_m^{1;total}$, as defined in (B.87), vanishes for small values of the index m. Due to (B.92) this expression then vanishes identically for all values of the index m.

Appendix C

In this appendix we elaborate on the arguments given below (3.10). Let $\phi(z)$ be a Kač-degenerate primary field, such as e.g. $A_{5/8}(z)$ in (3.10). Let Λ be a polynomial in Virasoro lowering operators L_{-m} $(m \geq 1)$, so that

$$\big(\Lambda\phi\big)(0) \ \text{ is Virasoro primary,} \quad \text{i.e } \ L_{+n}\big(\Lambda\phi\big)(0) = 0, \quad (n \geq 1)\,. \tag{C.1}$$

An example is $\Lambda = \big(L_{-2} - \frac{2}{3}L_{-1}L_{-1}\big)$ in Eq. (3.10).

Assume that $\big(\Lambda\phi\big)$ can be set to zero when inserted into any correlation function with other operators. This implies in particular that

$$\langle\big(\Lambda\phi\big)(0)\ \phi_1(z_1)\ \phi_2(z_2)\ \phi_3(z_3)...\phi_N(z_N)\rangle = 0 \tag{C.2}$$

for all primary operators $\phi_1, ...\phi_N$. We will show that

$$\ell_{+n}\big(\Lambda\phi\big)(0) \neq 0 \tag{C.3}$$

for *some* $n \geq 1$ leads to a contradiction with (C.2). To see this, consider the special case where $\phi_2 = \phi_1^\dagger$ has non-vanishing two-point function $< \phi_1(z)\phi_1^\dagger(0) > \neq 0$. Thus, we know from (2.16) and (C.2) that

$$\langle\Lambda\phi(0)\ t(z_2)\ \phi_3(z_3)...\phi_N(z_N)\rangle = 0 \tag{C.4}$$

since the insertion of $T(z)$, appearing also in the OPE $\phi_1(z_1)\phi_1^\dagger(z_2)$, vanishes due to (C.2) because T is primary at central charge $c = 0$. Hence we conclude from the representation (3.3) of ℓ_n, where the integration contour surrounds the origin, that

$$\langle\ell_n\big(\Lambda\phi\big)(0)\ \phi_3(z_3)...\phi_N(z_N)\rangle = 0, \qquad (\text{for all } n \geq 1) \tag{C.5}$$

for all primary operators $\phi_3, ...\phi_N$ (again, due to (C.2) and because $T(z)$ is primary). This completes the proof.

Acknowledgments

We would like to thank Ian Kogan, John Cardy and Monwhea Jeng for illuminating discussion or correspondence. This work was supported in part by the NSF through Grant No. DMR-00-75064 (A.W.W.L).

References

1. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).
2. P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997).
3. H. W. J. Blöte, J. L. Cardy and M. P. Nightingale, Phys. Rev. Lett. **56**, 7462 (1986).
4. I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).
5. V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B **247**, 83 (1984.
6. See e.g.: I. Affleck and F. D. M. Haldane, Phys. Rev. B **36**, 5291 (1987).
7. See e.g.: I. Affleck and A. W. W. Ludwig, Nucl. Phys. B **360**, 641 (1991); A. W. W. Ludwig and I. Affleck Nucl. Phys. B **428**, 545 (1994).
8. M. Freedman, C. Nayak, K. Shtengel, K. Walker and Z. Wang, Ann. Phys. **310**, 428 (2004).
9. See e.g.: K. B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, UK, 1997).
10. See e.g.: A. J. McKane and M. Stone, Ann. Phys. **131**, 36 (1981).
11. D.-H. Lee and Z. Wang, Phys. Rev. Lett. **76**, 4014 (1996).
12. R. T. F. van Schaijk, A. de Visser, S. Olsthoorn, H. P. Wei and A. M. M. Pruisken, Phys. Rev. Lett. **84**, 1567 (2000).
13. B. Huckestein, Rev. Mod. Phys. **67**, 357 (1995).
14. For a review, see e.g. Ref. [13].
15. H. Levine, S. B. Libby and A. M. M. Pruisken, Phys. Rev. Lett. **51**, 1915 (1983).
16. D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. **38**, 454 (1983); JETP Lett. **38**, 552 (1983).
17. I. Gruzberg, A. W. W. Ludwig and N. Read, Phys. Rev. Lett. **82**, 4524 (1999).
18. E. J. Beamond, J. Cardy and J. T. Chalker, Phys. Rev. B **65**, 214301 (2002).
19. The conformal spectra of the SUSY theories for 2D percolation and dilute polymers were recently found in: N. Read and H. Saleur, *Nucl. Phys.* B **613**, 409 (2001).
20. G. Parisi and N. Sourlas, J. Physique Lett. (Paris) **41**, L403 (1980).
21. For a review, see e.g.: T. C. Lubensky, in *Les Houches Session XXXI*, eds. Balian, Maynard and Toulouse (North Holland, Amsterdam 1979).
22. P. G. deGennes, Phys. Lett. A **38**, 339 (1972).
23. See e.g.: B. Nienhuis, J. Stat. Phys. **34**, 731 (1984).
24. H. Saleur and B. Duplantier, Phys. Rev. Lett. **58**, 2325 (1987).
25. see e.g. F.Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
26. See e.g.: A. W. W. Ludwig, Nucl. Phys. B **330**, 639 (1990); Vl. S. Dotsenko, M. Pico and P. Pujol, Nucl. Phys. B **455**, 701 (1995); J. Cardy and J. L. Jacobsen, Phys. Rev. Lett. **79**, 4063 (1997); C. Chatelain and B. Berche, Nucl. Phys. B **572**, 626 (2000); T. Davis and J. Cardy, Nucl. Phys. B **570** 713 (2000); M. Jeng and A. W. W. Ludwig, Nucl. Phys. B **594**, (2001).
27. V. Gurarie and A. W. W. Ludwig, J. Phys. A: Math. Gen. **35**, L377 (2002).
28. see e.g. Appendix B of Ref. [1], or Section B.1 of Appendix B of this paper.
29. V. Gurarie, Nucl. Phys. B **546**, 765 (1999).
30. J. L. Cardy, *The Stress Tensor in Quenched Random Systems*, in *Proceedings of the NATO Advanced Research Workshop on Statistical Field Theories*, Como 2001 (Kluwer, Boston, 2002) [http://www-thphys.physics.ox.ac.uk/users/JohnCardy/seminars/como.pdf].
31. Monwhea Jeng, private communication.
32. V. Gurarie, Nucl. Phys. B **410**, 535 (1993).
33. J.-S. Caux, I. I. Kogan and A. M. Tsvelik, Nucl. Phys. B **466**, 444 (1996).

34. I. I. Kogan and A. Nichols, Int. J. Mod. Phys. A **18**, 4771 (2003).
35. J. L. Cardy, *Logarithmic Correlations in Quenched Random Magnets and Polymers*, cond-mat/9911024.
36. We thank Monwhea Jeng, who has checked [31] these statements up to level 15, for assistance.
37. A. B. Zamolodchikov, Mod. Phys. Lett. A **6**, 1807 (1991).
38. A. B. Zamolodchikov, in *Advanced Studies in Pure Mathematics* **19**, K. Aomoto and T. Oda eds. (Academic Press, New York, 1989).
39. L. Chim and A. B. Zamolodchikov, Int. J. Mod. Phys. A **7**, 5317 (1992).
40. P. Fendley and N. Read, J. Phys. A **35**, 10675 (2002).
41. Vl. S. Dotsenko, Nucl. Phys. B **235** (1984) 54; Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B **240** (1984) 312.
42. J. L. Cardy, J. Phys. A **25**, L201 (1992).
43. A. B. Zamolodchikov and A. Fateev, Sov. Phys. JETP **62**, 215 (1985) [Zh. Eksp. Teor. Fiz. **89**, 380 (1985)].
44. I. I. Kogan and A. Nichols, *Stress Energy Tensor in* $c = 0$ *Logarithmic Conformal Field Theory*, hep-th/0203207.