FLUID DYNAMICS OF NSR STRINGS

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We show that the renormalization group flows of massless superstring modes in the presence of fluctuating D-branes satisfy the equations of fluid dynamics. In particular, we show that the D-brane's U(1) field is related to the velocity function in the Navier-Stokes equation while the dilaton plays the role of the passive scalar advected by the turbulent liquid. This leads us to suggest a possible isomorphism between the off-shell superstring theory in the presence of fluctuating D-branes and fluid mechanical degrees of freedom.

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This work is dedicated to the memory of Ian Kogan, my collaborator and wonderful friend. Certain initial ideas, which eventually evolved into some of the results of this work, were discussed almost a year ago in our last phone talk with Ian, right before his departure from Oxford to Trieste. We had a productive conversation and agreed to continue the discussion few days later, after Ian's supposed return. Ian tragically passed away in Trieste on the 4th of June, 2003.

1. Introduction

It is common to think of strings as objects existing only at some exotically small scales, such as the Planck scale; however in reality it is not necessarily so. Perhaps the best-known example of a non-Planckian string is a confining (QCD) string, consisting of gluon field lines confined to narrow flux tubes connecting a pair of quarks [1],[2]. These thin flux tubes, whose typical sizes are of the order of Λ_{QCD} are actually nothing but the objects that we call "confining strings". This means that one can think of an isomorphism, or a glossary, between string and gauge-theoretic degrees of freedom. The purpose of this paper is to point out and explore another example where stringy structures arise. Namely, we will show how the equations of hydrodynamics (e.g. Navier-Stokes and the passive scalar [3] equations, etc) appear in the dynamics of strings in fluctuating D-brane-like backgrounds. In particular, these results may imply that one could attempt to use the language of string theory to study the dynamics of vortices in a turbulent flow and, hopefully, to draw some important conclusions about the physics of turbulence from the the string-theoretic formalism. Clearly, the scale at which these "hydrodynamical" strings would live could be of the order of the lower bound of the inertial range of the fluid dynamics.

Before we start the calculations, let us briefly return to the QCD strings. The problem of the gauge-string correspondence has drawn a lot of attention in recent years and significant progress has been made (an especially well-known example is the AdS/CFT correspondence observed in the supersymmetric case) [4],[5],[6]. At this point, the string-theoretic approach remains among the most promising in our attempts to understand large N QCD dynamics in four dimensions.

The concept of the gauge-string correspondence particularly predicts that open QCD strings, connecting the quarks in the four-dimensional gauge theory, actually live in the curved 5-dimensional space-time AdS_5. The AdS_5 geometry of our space-time is the consequence of the condition that the partition function $Z(C)$ of an open string with its ends attached to some contour C in four dimensions is annihilated by the 4-dimensional loop operator, i.e. we can identify $Z(C)$ with the expectation value of the corresponding Wilson loop in QCD [2]. This means that the problem of the gauge-string correspondence is equivalent to studying the dynamics of strings in curved backgrounds. Needless to say, apart from the particular problem of the gauge-string correspondence, developing a consistent theory of strings and D-branes in curved backgrounds is crucial for understanding the non-perturbative aspects of M-theory dynamics and string dualities. In general, string dynamics in curved backgrounds is nonlinear and complicated and we have little to say about the spectrum, correlation functions etc. Our understanding of D-branes is also incomplete, with many results being based on arguments coming from the low-energy effective theory. A second quantized formalism for D-branes, in the form that has been developed for perturbative string theory, is still lacking. Some time ago it was realized that in the spectrum of covariant NSR superstring theory there exists a class of physical vertex operators which can be interpreted as second-quantized creation operators for D-branes and used to explore the non-perturbative dynamics of strings and D-branes in curved backgrounds, while technically still working with perturbative string amplitudes in flat space-time; in other words, these vertex operators allow us to study strings and D-branes in curved backgrounds by using an essentially background independent formalism. These vertex operators are BRST invariant nontrivial massless states in NSR superstring theory, which exist at nonzero superconformal ghost pictures only. We will refer to them as the vertices with ghost-matter mixing, or as brane-like states. Their unusual picture-dependence is due to the global singularities in the moduli spaces of the Riemann surfaces which are created by the insertions of these vertices on the worldsheet [7]. The picture independence of superstring amplitudes (allowing us to move picture changing operators inside the correlators, up to total derivatives in the moduli space) is thus violated by these singularities. The global singularities lead to the appearance of boundaries with non-trivial topology in the space of the supermoduli, and the total derivatives become important, causing the picture dependence. In the open string case, the simplest examples of these vertices are the massless NS 5-forms which exist at pictures $+1$ and -3; in the low energy limit these vertices lead to the RR couplings of the D-branes in the DBI effective action. The closed string brane-like states are in turn related to the kinetic (sqrt of the determinant) terms of the DBI action. Thus the brane-like states can be understood as D-brane creation vertices in the second quantized formalism. For this reason considering these vertices as sigma-model terms in the NSR string theory is equivalent to introducing D-branes, or curving the background. Thus D-brane dynamics can be reformulated in terms of scattering amplitudes involving brane-like vertex operators; the scattering amplitudes of the usual perturbative string states with brane-like insertions will reproduce the correlators of a photon, a graviton or a dilaton in the curved backgrounds created by these D-branes.

As we shall see, the non-perturbative character of the brane-like states is encrypted in their non-trivial picture dependence as well as the picture dependence of their OPE coefficients. As the insertions of the brane-like states are equivalent to creating D-branes and curving the backgrounds, it is important to understand the mechanism of how the curving occurs. For instance, one problem of interest is as follows. Suppose that at the initial moment of time we have a dilaton plane wave propagating in the initially flat space-time. At a certain moment of time a D-brane is inserted in the background; e.g. it can be created by the relevant second quantized vertex operator. The insertion of a D-brane leads to strong fluctuations of the metric which later stabilizes to some static brane-like geometry. Then how would the dilaton wave evolve in this process of curving? Another question of interest is the behavior of the D-brane's U(1) field in this process. As we will see in this paper, attempting to investigate these questions leads to hydrodynamical equations for the RG flows of the massless superstring modes.

In terms of the worldsheet CFT, the curving of space-time corresponds to the RG flow from a conformal point describing strings in Minkowski space to a new conformal point corresponding to the worldsheet CFT of strings in the curved brane-like geometry. Therefore one has to study the β functions of the open and closed string states in the presence of the ghost-matter mixing vertex operators. It turns out that, because of the ghost-matter mixing, these β functions are crucially different from the usual ones in string theory; namely these β functions become stochastic, i.e. contain manifest dependence on the brane-like vertex operators, which are the worldsheet variables. From the point of view of the space-time fields entering the β function equations the vertex operators inserted on the worldsheet are the random variables; they can be understood as sources of non-Markovian stochastic noise, whose memory structure is determined by the cutoff dependence of the correlators of the ghost-matter mixing vertices. Moreover we find that the stochastic β function equations for the dilaton and the photon are nothing but the equations of hydrodynamics; we obtain the passive scalar equations for the dilaton and the Navier-Stokes equation for the stochastic vector quantity which is the D-brane U(1) field times the D-brane creating vertex operator. The incompressibility of this "superstringy fluid" follows just from the transversality of the photon.

This paper is organized as follows. In the first section, we review the construction of the sigma-model with the brane-like states, starting from the integration over the moduli and supermoduli of gravitini. The ghost-matter mixing phenomenon and existence of the picture-changing anomalies will follow from the singularities in the moduli space. In the second section, we calculate the β functions of the photon and the dilaton and show they lead to the stochastic equations of hydrodynamics – Navier-Stokes and the passive scalar equations respectively. In the concluding section we discuss possible physical implications of our results.

2. NSR sigma-model and Conformal β functions

Consider the superstring theory in NSR formalism, perturbed by the set $\{V_i,\, i=1,...,n\}$, of massless physical vertex operators (from both the open and closed string sectors, e.g. a photon and a dilaton). The sphere or disc scattering amplitude for N vertex operators in the NSR superstring theory is given by [8], [9], [7]

$$
\begin{aligned}
&\langle V_1(z_1,\bar{z}_1)...V_N(z_N,\bar{z}_N)\rangle \\
&=\int\prod_{i=1}^{M(N)} dm_i d\overline{m_i}\int\prod_{a=1}^{P(N)} d\theta_a d\overline{\theta_a}\int DXD\psi D\overline{\psi}D[ghosts] \\
&\times \mathrm{e}^{-S_{NSR}+m_i\langle\xi^i|T_m+T_{gh}\rangle+\overline{m_i}\langle\overline{\xi^i}|\overline{T}_m+\overline{T}_{gh}\rangle+\theta_a\langle\chi^a|G_m+G_{gh}\rangle+\overline{\theta}_a\langle\overline{\chi}_a|\overline{G}_m+\overline{G}_{gh}\rangle} \\
&\times\prod_{a=1}^{M(N)}\delta(\langle\chi^a|\beta\rangle)\delta(\langle\overline{\chi}^a|\overline{\beta}\rangle)\prod_{i=1}^{P(N)}\langle\xi^i|b\rangle\langle\overline{\xi}^i|\overline{b}\rangle V_1(z_1,\bar{z}_1)...V_N(z_N,\bar{z}_N).
\end{aligned}
\tag{2.1}
$$

Here $z_1,...,z_N$ are the points of the vertex operator insertions on the sphere and

$$
S_{NSR}\sim\int d^2z\{\partial X_m\overline{\partial}X^m+\psi_m\overline{\partial}\psi^m+\overline{\psi}_m\partial\overline{\psi}^m+b\overline{\partial}c+\overline{b}\partial\overline{c}+\beta\overline{\partial}\gamma+\overline{\beta}\partial\overline{\gamma}\} \\
m=0,...,9
\tag{2.2}
$$

is the NSR superstring action in the superconformal gauge where (m_i,θ_a) are the holomorphic even and odd coordinates in the moduli superspace and (ξ^i,χ^a) are their dual super Beltrami differentials (similarly for $(\overline{m_i},\overline{\theta}_a)$ and $(\overline{\xi}^i,\overline{\chi}^a)$). The $\langle...|...\rangle$ symbol stands for the scalar product in the Hilbert space and the delta functions $\delta(\langle\chi^a|\beta\rangle)$ and $\delta(\langle\xi^i|b\rangle) = \langle\xi^i|b\rangle$ are needed to ensure that the basis in the moduli space is normal to variations along the superconformal gauge slices (similarly for the antiholomorphic counterparts) [8], [10]. The dimensionalities $M(N)$ and $P(N)$ of the even and odd supermoduli spaces depend on the number N of vertex operator insertions and are given by [7]

$$\begin{aligned} M(N) &= N - 3\,, \\ P(N) &= N_{NS} + \frac{1}{2}N_R + 3N_b - 2\,, \\ N &\equiv N_{NS} + N_R + N_b\,, \end{aligned} \tag{2.3}$$

where N_{NS}, N_R and N_b are the number of NS, Ramond and brane-like vertex operator insertions respectively in the N-point amplitude. As has been shown in [7] the expression (2.1) for the amplitudes leads to the following generating functional for the NSR sigma-model,

$$Z(\varphi_i) = \int DXD\psi D\overline{\psi}D[ghosts]\,\mathrm{e}^{-S_{NSR}+\int_k \varphi_i(k)V^i(k,z_i)}\rho_{(\Gamma;Z)}\,\rho_{(\overline{\Gamma};\overline{Z})}\,, \tag{2.4}$$

where φ_i are the space-time fields corresponding to the V_i vertices and $\rho_{(\Gamma,Z)}$ and its complex conjugate $\rho_{(\overline{\Gamma},\overline{Z})}$ [11] are the picture-changing factors necessary to ensure the cancellations of the $\beta-\gamma$ and $b-c$ ghost anomalies (equal to 2 for the $\beta-\gamma$ and 3 for the $b-c$ systems). In the sigma-model (2.2) all the vertex operators are to be taken in the unintegrated form, at the superconformal pictures (-1) or $(-1,-1)$ for the perturbative NS or NS-NS vertices (such as a photon or a dilaton), $(-1/2)$ for the Ramond vertices and (-3) for the brane-like vertices with ghost-matter mixing. The choice of the insertion coordinates z_i for unintegrated vertices is related to the choice of the Koba-Nielsen measure [12]. As was shown in [7] such a choice of ghost pictures is dictated by the number of independent quadratic and 3/2-differentials corresponding to the vertex operator insertions, and is consistent with the ghost number anomaly cancellation conditions for the fermionic and bosonic ghosts. The BRST invariant picture-changing operators for the $\beta-\gamma$ and the $b-c$ system are given by [13],[7]

$$\begin{aligned} :\Gamma &:= \delta(\langle chi^a|\beta\rangle)\langle\chi|G\rangle\,, \\ :Z &:= \langle\xi^i|b\rangle\,\delta(\langle\xi^i|T\rangle)\,, \\ i = 1...P(N);&\quad a = 1...M(N)\,, \end{aligned} \tag{2.5}$$

where χ^a and ξ^i are the basic vectors in the spaces of super Beltrami differentials, dual to the odd and even superconformal moduli. In the delta-functional basis for $\xi^i = \delta(z - z_i)$ and $\eta^a = \delta(z - z^a)$, the bosonic and fermionic picture-changing operators can be expressed as

$$\begin{aligned}
\Gamma(z_a) &=: \mathrm{e}^{\phi}(G) : (z_a)\,,\\
Z_{open}(z_i) &= \oint \frac{dw}{2i\pi}(w - z_i)^3 R(w) Z_{closed}(z_i, \overline{z}_i) = \int d^2w |w - z_i|^6 R(w)\overline{R}(\overline{w})\,,\\
R(w) &= bT(w) - 4c\,\mathrm{e}^{2\chi - 2\phi}\,TT(w) - 4b\,c\,\partial c\,\mathrm{e}^{2\chi-2\phi}\,T(w)\,.
\end{aligned} \tag{2.6}$$

Here G and T are the full worldsheet matter+ghost supercurrent and stress tensor and ϕ and χ are the bosonized ghost fields for the $\beta - \gamma$ system. The picture-changing operators for the $b - c$ system, Z_{open} and Z_{closed} change the $b - c$ ghost numbers of physical vertex operators by 1 unit and particularly map the unintegrated open or closed string vertices (of zero conformal dimension) to vertices of dimension 1 and $(1, 1)$ respectively, integrated over the worldsheet boundary or the entire worldsheet. It is convenient to choose z_i and z_a at the insertion points of the vertex operators, corresponding to the orbifold points in the spaces of supermoduli. In this case, the symmetry related to the picture-changing is reduced to the discrete automorphism group with finite volume. This automorphism group includes all the possible permutations of picture-changing operators between the insertion points of the vertices inside correlation functions [7]. The precise expression for $\rho_{(\Gamma,Z)}$ in terms of the basic vectors in the spaces of super Beltrami differentials is given by [7]

$$\begin{aligned}
\rho_{(\Gamma;Z)} &= \sum_{m,n=0}^{\infty} \Xi_{\xi}^{-1}(n)\Xi_{\chi}^{-1}(m)\\
&\times \sum_{\{\xi^{(1)},..\xi^{(n)},\chi^{(1)}...\chi^{(m)}\}} \delta(\langle\chi^{(1)}|\beta\rangle)\langle\chi^{(1)}|G\rangle...\delta(\langle\chi^{(m)}|\beta\rangle)\langle\chi^{(m)}|G\rangle\\
&\times \langle\xi^{(1)}|b\rangle\delta(\langle\xi^{(1)}|T\rangle)...\langle\xi^{(n)}|b\rangle\delta(\langle\xi^{(n)}|T\rangle)
\end{aligned} \tag{2.7}$$

and similarly for $\rho_{(\overline{\Gamma};\overline{Z})}$ where the sum over $\xi^{(i)}$ and $\chi^{(i)}$ implies the summation over all the basic vectors of the (m, n)-dimensional spaces of the differentials; Ξ_χ and Ξ_ξ are the volumes of the automorphism groups corresponding to the permutations of Γ and Z picture-changing operators between the vertices. These volumes factorize. For example, for the scattering amplitude of N gravitons on the sphere one has $\Xi_\xi = N^{N-3}$, $\Xi_\chi = N^{N-2}$. Structurally, it is easy to see from (7) that the ρ-factor has the form $\rho_{(\Gamma,Z)} \sim (1+ : \Gamma : + : \Gamma^2 : + : \Gamma^3 : +...)(1+ : Z : + : Z^2 : + : Z^3 : +...)$.

As the pictures of the vertex operators in the sigma-model (2.4) are fixed from the beginning, the role of the ρ-factor is to ensure the ghost number anomaly cancellation in the amplitudes. For example, in the case of the scattering of N unintegrated gravitons on the sphere the only term in the $\rho_{(\Gamma,Z)}$-expansion contributing to the correlator is of the structure $\sim :\Gamma^{N-2}Z^{N-3}:$, while contributions from other terms are absent due to the ghost number balance. In other words, each N-point amplitude automatically picks up the appropriate terms from the ρ factor to insure the correct overall ghost number and the factorization condition. Dividing by the number of permutations of the picture-changing operators between the vertices (equal to the volume of the automorphism group for a particular amplitude) leads to the appropriate normalization of the correlator and the corresponding terms in the low-energy effective action. Thus in the case of the usual perturbative vertices the $\rho_{(\Gamma,Z)}$ factor simply ensures the correct normalization and ghost anomaly anomaly cancellation in the amplitudes and the conformal β functions; in the picture dependent case of the brane-like vertices (corresponding to global singularities of the moduli spaces) things become more subtle and complicated.In the next section we will demonstrate how the picture-dependence of the OPE coefficients leads to the stochastic terms in the stochastic RG equations.

3. Picture dependent OPE and Stochastic β functions

Consider the expansion of the generating functional (2.4) of the sigma-model in terms of the space-time fields φ_i. We get

$$\begin{aligned} Z = \int D[X,\psi,\overline{\psi},ghosts]\mathrm{e}^{-S_{NSR}}\rho_{(\Gamma,Z)}\rho_{(\overline{\Gamma},\overline{Z})} \\ (1+\varphi_i V_i + \frac{1}{2}\varphi_i\varphi_j V_i V_j + \frac{1}{6}\varphi_i\varphi_j\varphi_k V_i V_j V_k + ...), \\ i,j,k = 1,...n\,. \end{aligned} \tag{3.1}$$

The Z transformation of the vertices (by the appropriate expansion terms from $\rho_{\Gamma,Z}$ and its complex conjugate) leads to [7]

$$\begin{aligned} :Z\overline{Z}V_i:\sim \int d^2z\, W_i(z,\overline{z}) + \{Q_{brst},...\}\,, \\ i = 1,...n\,, \end{aligned} \tag{3.2}$$

for the closed string vertices and

$$\begin{aligned} :Z\overline{Z}V_i:\sim \int \frac{d\tau}{2i\pi}\, W_i(\tau) + \{Q_{brst},...\}\,, \\ i = 1,...n\,, \end{aligned} \tag{3.3}$$

for open strings. The W_i are the dimension 1 or (1,1) operators which worldsheet integration gives as the integrated vertices for open and closed strings. Now consider the quadratic term in the expansion of (3.1) which, upon Fourier transform, is proportional to

$$\int_{p,q} \varphi_i(p)\varphi_j(q) \int d^2z \int d_2w W_i(z,\overline{z};p)W_j(w,\overline{w};q)\,, \qquad (1)$$

where $\int_{p,q}$ stands for the integral over the momenta of the vertices, p and q; the open string case can be treated analogously. In general, the OPE between W_i and W_j is singular and is given by.

$$W_i(z,\overline{z};k)W_j(w,\overline{w};p) \sim \frac{1}{|z-w|^2}\, C^k_{ij}(p;q)\, W_k\Big(\frac{z+w}{2}, \frac{\overline{z}+\overline{w}}{2}; p+q\Big) + ...\,, \qquad (3.4)$$

where C^k_{ij} are the structure constants and the summation over k is from 1 to n (i.e. over all the physical massless vertex operators). This OPE singularity leads to the divergence in the partition function as

$$\begin{aligned} &\int_{p,q} \int d^2z \int d^2w\, W_i(z,\overline{z};p)W_j(w,\overline{w};q) \\ &\sim \int_{k_1,k_2} C^k_{ij}(k_1,k_2) \int \frac{d^2\eta}{|\eta|^2} \int d^2\xi\, W_k(\xi,\overline{\xi};k_1) \\ &\equiv \log\Lambda \int_{k_1,k_2} C^k_{ij}\Big(\frac{k_1+k_2}{2}, \frac{k_1-k_2}{2}\Big) \int d^2\xi\, W_k(\xi,\overline{\xi};k_1)\,, \end{aligned} \qquad (3.5)$$

where Λ is the worldsheet cutoff and

$$\begin{aligned} k_1 &= p+q\,, \\ k_2 &= p-q\,, \\ \xi &= \frac{z+w}{2}\,, \\ \eta &= \frac{z-w}{2}\,. \end{aligned} \qquad (3.6)$$

To eliminate this divergence one has to renormalize the space-time field φ_k in the linear term in the expansion (3.1) as

$$\varphi^k(p) \to \varphi^k(p) - \frac{1}{2}\log\Lambda \int_q C^k_{ij}\Big(\frac{p+q}{2}, \frac{p-q}{2}\Big)\varphi_i\Big(\frac{p+q}{2}\Big)\varphi_j\Big(\frac{p-q}{2}\Big). \quad (3.7)$$

The conformal invariance condition on the worldsheet (i.e. the cutoff independence of the partition function) implies the low-energy effective equations of motion for the space-time fields

$$\int_q C_{ij}^k\Big(\frac{p+q}{2},\frac{p-q}{2}\Big)\varphi_i\Big(\frac{p+q}{2}\Big)\varphi_j\Big(\frac{p-q}{2}\Big)=0\,. \tag{3.8}$$

This particularly implies Einstein's equations for the closed string massless fields and the DBI equations of motion for the photon [14]. The renormalization group flow (3.7) generated by the OPE singularity (3.5) also deforms the next order quadratic term in the expansion (3.1). Under the renormalization (3.7) the quadratic term flows into the divergent term cubic in φ as

$$\begin{aligned}&\frac{1}{2}\int_{k,p}\varphi_i(k)\varphi_j(p)\int d^2z\int d^2w\,W_i(z,\overline{z})W_j(w,\overline{w})\\&\to-\frac{1}{2}\log\Lambda\int_{k,p,q}C_{ij}^k\Big(\frac{p+q}{2},\frac{p-q}{2}\Big)\varphi_i(k)\varphi_j\Big(\frac{p+q}{2}\Big)\varphi_k\Big(\frac{p-q}{2}\Big)\\&\qquad\times\int d^2z\int d^2w\,W_i(z,\overline{z};k)W_j(w,\overline{w};p).\end{aligned} \tag{3.9}$$

This divergent term, however, is precisely canceled by the divergence stemming from the OPE singularities inside the cubic term of the expansion (3.1). Indeed, using the OPE (3.4) it is easy to show that

$$\begin{aligned}&\frac{1}{6}\int_{k,p,q}\varphi_i(k)\varphi_j(p)\varphi_k(q)\int d^2z\int d^2w\int d^2u\,V_i(z,\overline{z};k)V_j(w,\overline{w};p)V_k(u,\overline{u};q)\\&\to\frac{1}{2}\log\Lambda\int_{k,p,q}C_{ij}^k\Big(\frac{p+q}{2},\frac{p-q}{2}\Big)\varphi_i(k)\varphi_j\Big(\frac{p+q}{2}\Big)\varphi_k\Big(\frac{p-q}{2}\Big)\\&\times\int d^2z\int d^2w\,W_i(z,\overline{z};k)W_j(w,\overline{w};p)\,.\end{aligned} \tag{3.10}$$

The cancellation of the divergences (3.9) and (3.10), stemming from the flow of the quadratic term and the OPE singularities of the cubic term, is important as it ensures that the RG equations for the space-time fields φ_i are determinnistic. Indeed, the presence of divergences of the type $\sim \log\Lambda\,\varphi_i\varphi_j\varphi_k {C_{ij}}^k\int W_iW_j$ in the expansion would lead to the appearance of extra terms in conformal β functions for the space-time fields $\frac{d\varphi_i}{d\log\Lambda}$, proportional to $\sim C_{jk}^l\varphi_i\varphi_j\varphi_k\int_\Lambda V_l$ with the worldsheet integral of the V_l vertex operators cut off at the scale Λ. In other words, the RG equations for the space-time fields would manifestly depend on the worldsheet variables! From the point of view of the cutoff dependent space-time physics, the vertex operators, involving the random functions defined on the worldsheet (such as the coordinate $X(z,\overline{z})$) play the role of the non-Markovian stochastic noise.

Indeed, the β function equations for the space-time fields would have the form of the Langevin stochastic equation

$$\frac{d\varphi}{d\tau} = \frac{\delta S(\varphi)}{d\varphi} + C\varphi^3\eta(\tau) \tag{3.11}$$

where C are the structure constants, the stochastic time variable $\tau \equiv \log\Lambda$ is defined by the logarithm of the worldsheet cutoff and the role of the stochastic noise is played by the worldsheet integral of V, cut off at a scale Λ,

$$\eta \equiv \int_\Lambda d^2 z V(z,\overline{z}) \,. \tag{2}$$

The memory structure of the noise is defined by the worldsheet correlations of the V operators. In any case, as we have already noted above, stochastic terms of the form (3.11) do not appear in the β function equation in standard string perturbation theory as the divergences (3.9) and (3.10) cancel each other. At first glance, the cancellation of (3.9) and (3.10), ensuring the determinism of the worldsheet β functions, is quite trivial and automatic, true for all orders of the perturbative expansion. Things, however, become far more subtle for the cases when the structure constants are picture dependent, i.e. the OPE $V_i V_j \sim C V_k$ depends on the ghost pictures at which V_i and V_j are taken. For perturbative vertices, such as a photon or a graviton, such a picture dependence is of course absent and accordingly their β functions are deterministic. The situation is drastically different in the case of ghost-matter mixing, i.e. in the presence of the vertex operators breaking the equivalence of the ghost pictures [7], [11].

The space-time fields corresponding to these vertices usually correspond to the collective coordinates of D-branes – their effective actions are shown to be of the DBI type; one can also show that these operators carry nonzero RR charges by computing their correlators with the RR vertices [15]. In fact, these vertex operators can be thought of as D-branes in the second-quantized formalism [15]. Important examples of such operators (at the integrated $b-c$ picture) are given by

$$\begin{aligned} V_5^{o.s.(-3)} &= R_{m_1...m_5}(k)\oint \frac{d\tau}{2i\pi} \mathrm{e}^{-3\phi}\psi_{m_1}...\psi_{m_5}\mathrm{e}^{ikX}\,, \\ V_5^{o.s.(+1)} &= R_{m_1...m_5}(k)\oint \frac{d\tau}{2i\pi} \mathrm{e}^{\phi}\psi_{m_1}...\psi_{m_5}\mathrm{e}^{ikX} + ghosts\,, \end{aligned} \tag{3.12}$$

for open strings and

$$V_5 \equiv V_5^{c.s(-3)} = H_{m_1...m_6}(k)\int d^2z e^{-3\phi-\overline{\phi}}\psi_{m_1}...\psi_{m_5}\overline{\psi}_{m_6}e^{ikX}(z,\overline{z}),$$
$$V_5^{c.s(+1)} = H_{m_1...m_6}(k)\int d^2z e^{\phi+\overline{\phi}}\psi_{m_1}...\psi_{m_5}\overline{\psi}_{m_6}e^{ikX}(z,\overline{z}) + ghosts, \tag{3.13}$$

for closed ones. Here and elsewhere the upper indices in the round brackets label the $\beta-\gamma$ ghost pictures. The open string V_5 operators exist at pictures -3 and below as well as $+1$ and above, but not at pictures -2, -1 and 0. In cases when the V_5 operators are taken at positive pictures ($+1$ and above), they must include the extra $b-c$ ghost terms to ensure their BRST invariance (apart from the $e^{\phi}\psi^5$ part) (see [11] or [7] for the precise expressions for these terms). All the higher positive pictures for V_5 operators can be obtained by applying the standard picture-changing to $V_5^{(+1)}$ while the lower negative pictures for V_5 can be derived using the inverse picture-changing of $V_5^{(-3)}$. $V_5^{(+1)}$ can be obtained from $V_5^{(-3)}$ by making the Z transformation first (i.e. putting $V_5^{(-3)}$ in the "double-integrated" form) and the subsequent application of the fourth power of the usual picture-changing operator. The closed-string V_5 operators (3.13) can be obtained straightforwardly from the open string ones by multiplying them by the antiholomorphic photonic part (which of course can be taken at any picture). The BRST invariance and non-triviality conditions for the operators (3.12), (3.13) are given by [15], [7]

$$k_{[m_6}R_{m_1...m_5]}(k) \neq 0,$$
$$k_{[m_7}H_{m_1...m_5]m_6}(k) \neq 0, \tag{3.14}$$
$$k_{m_6}H_{m_1...m_5m_6}(k) = 0,$$

where the H-tensor is antisymmetric in the first 5 indices and the square brackets imply antisymmetrization. In particular, the BRST conditions (3.14) can be shown to reduce the number of independent components of the H-tensor by one half, i.e. the number of the physical d.o.f. related to the H-tensor is equal to 1260. In addition, these BRST conditions imply that for each particular polarization $m_1...m_6$ of the H-tensor the momentum k must be orthogonal to the $m_1, ..., m_6$ directions in space-time. As the number of independent polarizations is given by $\frac{10!}{4!6!} = 210$, the number of physical degrees of freedom per polarization is equal to 6. By computing the low-energy effective action for the V_5-states with a fixed polarization one can show that these degrees of freedom correspond to 6 collective coordinates for a D3 brane's transverse fluctuations which can be parametrized as

$$\begin{aligned} \lambda_t \equiv H_{t_1 \ldots t_5 t} \, , \\ t = 4, ..., 9; \quad t \neq t_1, ..., t_5 \end{aligned} \tag{3.15}$$

(H is antisymmetric in $t_1, ..., t_5$) and the low-energy limit is described by the DBI action for a D3-brane with its world-volume in the $0, ..., 3$ directions, in terms of λ_t [7],[15]. In the spherically symmetric case (s-wave approximation in the near-horizon limit) one can take all the components equal

$$\lambda_t(k) \equiv \lambda(k); \qquad t = 4, ...9 \, . \tag{3.16}$$

To simplify things in the rest of this paper, the V_5-excitation entering the NSR sigma-model will be taken in the form (3.16). As the OPE's involving these operators are picture dependent (see the discussion below) one must accurately account for the β function contributions from various ghost pictures in the expansion (3.1). Then, as a result of the picture asymmetry in the operator products, the renormalization of the cubic terms won't necessarily cancel the flow of the quadratic ones, as in the case of (3.9) and (3.10). As a result, the resulting β function equations would be operator-valued, or stochastic. In particular, in the absence of the U(1) gauge field, this stochastic process leads to the evolution of the space-time geometry from flat to one with the D-brane metric corresponding to the appropriate DBI effective action (see the discussion above). In other words, the D-brane creation operators are inserted in the the theory with the background originally flat; such an insertion leads to non-Markovian stochastic process whose thermodynamical limit is given by the string theory in the curved D-brane type geometry. As a result, exploring the correlation function involving the brane-like states allows us to study the dynamics of strings in curved backgrounds, while technically working with string perturbation theory around the flat vacuum. Such an approach is referred to as the ghost-matter mixing formalism [11],[15]. In the presence of the U(1) gauge field, the stochastic β function equations, involving the brane-like states, become more complicated and take the form of the equations of hydrodynamics. The rest of this paper will discuss the derivation of these equation from the sigma-model for NSR strings, as well as the related physical implications.

4. β functions in D-brane Backgrounds and the Equations of Hydrodynamics

Consider the NSR superstring theory in the background of standard perturbative states (a photon, a dilaton,a graviton and an axion) and the closed string V_5 excitation (3.13), (3.15), (3.16) corresponding to the D3-brane insertion. For simplicity, let the polarization and the propagation of the perturbative states be confined to the four-dimensional subspace, defined by the admissible directions of the momentum of the V_5 vertex, which follow from the BRST conditions (3.14) (corresponding to the orientation of the D3-brane's world-volume).

The generating functional of this model is given by

$$\begin{aligned} Z(\lambda, \varphi, A_a, H_{ab}) = \int D[X, \Psi, \overline{\Psi}, ghosts] \exp\Big\{-S_{NSR} + \int_k \{A_a(k) V^a_{ph}(\tau; k) \\ + H^{ab} V_{ab}(z, \bar{z}; k) + \int_p \lambda(p) V_5^{(-3)}(w, \overline{w}; p)\Big\} \, |\rho_{(\Gamma, Z)}|^2 \,, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} V_{ph}(\tau) = c\, e^{-\phi} \psi^a e^{ikX}(\tau) \,, \\ V^{ab}(z, \bar{z}; k) = c\bar{c}\, e^{-\varphi - \overline{\varphi}} \psi^a \overline{\psi}^b e^{ikX}(z, \bar{z}) \,, \\ V_5^{(-3)}(w, \overline{w}; p) = c\bar{c}\, e^{-3\phi - \overline{\phi}} \psi_{[4} ... \psi_8 \overline{\psi}_{9]} e^{ip_a X^a}(w, \overline{w}) \,, \\ a = 0, ..., 3 \,, \end{aligned} \tag{4.2}$$

and the rank 2 massless field H_{ab} is either a dilaton, a graviton or an axion,

$$\begin{aligned} H_{ab} = G_{ab}(k) + B_{ab}(k) + \varphi(k)(\eta_{ab} - k_a \bar{k}_b - \bar{k}_a k_b) \,, \\ k^2 = \bar{k}^2 = 0; \quad (k\bar{k}) = 1 \,. \end{aligned} \tag{4.3}$$

Consider the expansion of this functional in the space-time fields. Namely, consider the term quadratic in λ and A_m, and of arbitrary order N in the closed string H field. This term is given by the correlation function

$$\begin{aligned} A(p_1, p_2; k_1, k_2; q_1, ..., q_N) \\ \sim \frac{\langle |\rho_{(\Gamma; Z)}|^2 V_5^{(-3)}(p_1) V_5^{(-3)}(p_2) V^a(k_1) V^b(k_2) V^{a_1 b_1}(q_1) ... V^{a_N b_N}(q_N) \rangle}{(2!)^2 N!}, \end{aligned} \tag{4.4}$$

where all the operators except for the V_5 vertices are at the left picture -1 and all the right pictures here and elsewhere are assumed to be -1 unless specified otherwise.

We need to find the contribution of this correlator to the photon's beta function in the presence of the V_5 insertions (i.e. in the D3-brane background). This contribution is determined by the OPE singularities between the V_5 operators and the photon vertices inside the correlator (4.4). Clearly, such a contribution must depend only on the structure constants of this operator product and not on any other insertions not involved in the OPE (in particular, the N-independence is the consequence of the factorization conditions for the string amplitudes, as well as the condition for the renormalizability of the string perturbation theory). In the case where the structure-constants are picture independent, such independence of N is quite obvious. This is why in the β function calculations (3.7), (3.9), (3.10) it is sufficient to consider the OPE of a couple of the vertex operators V_i and V_j regardless of the details of the particular correlators they belong to – normalization and unitarity conditions would always ensure that the RG flow (3.7) of the space-time fields would simultaneously cancel the divergences stemming from the OPE (3.4) inside all the correlators. In the picture dependent case involving the V_5 operators the situation is far more subtle and perturbation theory must be modified in order to satisfy the conditions of renormalizability and factorization of amplitudes. Let us now concentrate on the scattering amplitude (4.4). The relevant term from $\rho_{(\Gamma,Z)}\rho_{(\overline{\Gamma Z})}$ to cancel the left and right ghost number anomalies has the structure $\sim$: $(Z\overline{Z})^{N+1} :: \Gamma^{N+6} : : \overline{\Gamma} :^{N+2}$. The volumes of the automorphism groups related to the left and right Γ and Z permutations between the insertion points of the $(N+4)$-point correlator (4.4) with two V_5 vertices are computed to be

$$\begin{aligned}
\Xi_\Gamma(N+4;2) &= (N+4)^{N+6} \\
&- 2(N+3)^{N+6}\left\{\frac{N+6}{N+3} + \frac{(N+5)(N+6)}{2(N+3)^2} + \frac{(N+4)(N+5)(N+6)}{6(N+3)^3}\right\} \\
&+ (N+2)^{N+7}\left\{\frac{(N+5)(N+6)}{(N+2)^2} + \frac{(N+4)(N+5)(N+6)}{(N+3)^3}\right. \\
&+ \frac{7}{12}\,\frac{(N+3)(N+4)(N+5)(N+6)}{(N+2)^4} \\
&\left.+ \frac{1}{36}\,\frac{(N+1)(N+3)(N+4)(N+5)(N+6)}{(N+2)^5}\right\}, \\
\Xi_{\overline{\Gamma}}(N+4;2) &= (N+4)^{N+2}, \\
\Xi_\xi(N+4;2) &= \Xi_{\overline{\xi}}(N+4;2) = (N+4)^{N+1}.
\end{aligned} \tag{4.5}$$

Factorizing by these volumes in the $|\rho_{(\Gamma,Z)}|^2$ factor ensures the correct normalization of the related terms in the low-energy effective action, as well as the amplitude factorization and unitarity conditions (after summing over all the admissible picture configurations for the vertices in the amplitude

(4.4), equivalent to the admissible choices of basis in the supermoduli space associated with this amplitude). If the V_5 operators were picture independent, the Ξ volumes for the $(N+4)$-point amplitude would be given simply by the number of permutations of the $N+1$ insertions of picture-changing Z operators and $N+2$ insertions of Γ at $N+4$ points, i.e. $(N+4)^{N+1}$ and $(N+4)^{N+2}$ respectively. However, because of the $\beta-\gamma$ picture-dependence of the V_5 insertions in the $(N+4)$-point function, while the Ξ_ξ volume is still unchanged, the Ξ_Γ volume related to the left Γ permutations must exclude the combinations leading to the non-existent 0, -1 and -2 pictures for any of the V_5 insertions (recall that the V_5 operators are originally taken at the picture $-3,-1$ while the rest are at the picture $-1,-1$). A simple calculation then leads to (4.5), ensuring that the factorization and unitarity conditions for the amplitude (4.4) are satisfied. Next, consider the OPEs between the V_5 and V^a contributing to the photon's β function. As has been noted above, these OPE's are picture dependent. For example, the OPE of the picture 0 photon with V_5 gives

$$\begin{aligned}
&\lim_{z,\bar{z}\to\sigma} V^{a(0)}(k,\tau)V_5^{(-3,-1)}(p,z,\bar{z})\\
&= (\partial X^a + i(k\phi)\psi^a)\,\mathrm{e}^{ikX}(\tau)\,\mathrm{e}^{-3\phi-\bar{\phi}}(\psi_{t_1}...\psi_{t_5}\bar{\psi}_{t_6}+c.c.)\,\mathrm{e}^{ipX}(z,\bar{z}) \qquad (4.6)\\
&\sim ip^a\Big(\frac{1}{\tau-z}+\frac{1}{\tau-\bar{z}}\Big)V_5(k+p) = 2ip^a(\tau-\sigma)^{-1}V_5(k+p)
\end{aligned}$$

and contributes to the photon's β function since it is singular as $z,\bar{z}\to\sigma$ where σ is some point on the worldsheet boundary, close to τ. At the same time, the integrand of the photon's vertex operator at the picture $+1$ is given by

$$\begin{aligned}
V^{a(+1)}(k,\tau) = &-\frac{1}{2}\,\mathrm{e}^{\phi}\,\mathrm{e}^{ikX}\Big\{\big((\psi\partial X)(\partial X^a)+i(k\psi)\psi^a\big)+\frac{1}{2}\,\partial^2\psi^a+\partial\psi^a P^{(1)}_{\phi-\chi}\\
&+\psi^a P^{(2)}_{\phi-\chi}+i(k\psi)\partial X^a P^{(1)}_{\phi-\chi}+i(k\partial\psi)\partial X^a+i(k\partial X)P^{(1)}_{\phi-\chi}\psi^a\\
&+i(k\partial^2 X)\psi^a+i(k\partial\psi)(k\psi)\psi^a+i(k\psi)\big(\partial^2 X^a+P^{(1)}_{\phi-\chi}\partial X^a\big)\Big\}\\
&-\frac{1}{4}\,b\,\mathrm{e}^{2\phi-\chi}\mathrm{e}^{ikX}P^{(2)}_{2\phi-2\chi-\sigma}\big(\partial X^a+i(k\psi)\psi^a\big),
\end{aligned} \qquad (4.7)$$

where we have defined

$$P^{(n)}_f = \frac{1}{n!}\,\mathrm{e}^{-f}\,\frac{d^n}{dx^n}\,\mathrm{e}^f \qquad (4.8)$$

for any $f(x)$. It is now easy to check that the OPE of the picture $+1$ photon with V_5 is non-singular,

$$\lim_{z,\bar{z}\to\sigma} V^{a(+1)}(k,\tau)V_5^{(-3,-1)}(z,\bar{z}) \sim O(|\tau-\sigma|^0), \tag{4.9}$$

and therefore does not contribute to the β function. In the general case, one can show that the picture dependent OPE between the photon and V_5 is given by

$$\begin{aligned}
&V_5^{(s_1)}(z,\bar{z};p)V^{a(s_2)}(\tau;k)\\
&\quad\sim i\Big(\frac{1}{z-\tau}+\frac{1}{\bar{z}-\tau}\Big)\,p^a\,C^{(s_1|s_1+s_2)}\,V_5^{(s_1+s_2)}\Big(\frac{z+\tau}{2},\frac{\bar{z}+\tau}{2};k+p\Big)+\ldots,\\
&V_5^{(s_1)}(z,\bar{z};k)V_5^{(s_2)}(w,\bar{w};p)\sim\frac{C^{(s_1|s_2)}}{|z-w|^2}((kp)V_\varphi^{(s_1+s_2)}\Big(\frac{z_1+z_2}{2},\frac{\bar{z}_1+\bar{z}_2}{2};k+p\Big)\\
&\qquad\qquad+\frac{2k_m}{z-\bar{z}+w-\bar{w}}\,V^{m(s_1,s_2)}(\tau;k+p))\,,
\end{aligned} \tag{4.10}$$

where we have skipped the non-singular part of the OPE. The s_i indices label the pictures of the vertex operators and the symmetric picture matrix $C^{(s|t)}$ reflects the picture dependence of the OPE coefficients in the singular terms. Namely, $C^{(s|t)}=0$ if either of the pictures $s,t=0,-1,-2$, and 1 otherwise. Let us now return to the amplitude (4.4). Summing over the admissible configurations of the basic vectors in the supermoduli space, and suppressing the $\rho_{(\Gamma,Z)}$ factor, we obtain

$$\begin{aligned}
&A(p_1,p_2;k_1,k_2;q_1,...,q_N)\sim\frac{1}{(2!)^2N!}\,\Xi_\Gamma^{-1}(N+4;2)\\
&\times\sum_{s_1,s_2,t_1,t_2,r_1,\ldots r_{n-1};s_{1,2}\neq 1,2,3}^{s_1+s_2+t_1+t_2+\sum_{j=1}^N r_j=N+6}\frac{(N+6)!}{s_1!s_2!t_1!t_2!r_1!...r_{N-1}!(N+6-s_1-s_2-t_1-t_2-\sum r)!}\\
&\qquad\times\langle V_5^{(s_1-3)}(p_1)V_5^{(s_2-3)}(p_2)V_a^{(t_1-1)}(k_1)V_b^{(t_2-2)}(k_2)\prod_{j=1}^N V_{a_jb_j}^{(r_j-1)}(q_j)\rangle\,,
\end{aligned} \tag{4.11}$$

where we suppressed the antiholomorphic picture indices for the closed string operators (note that the antiholomorphic part of the amplitude is picture independent and the effect of the $\rho_{(\bar{\Gamma},\bar{Z})}$ factor is trivial; summing over pictures and dividing by the automorphism volume related to the antiholomorphic pictures results in the trivial unit normalization factor for the right-moving part of the amplitude, as in any picture independent case). The sum (4.11) gives the total contribution to the scattering amplitude (4.4) from all admissible picture configurations.

To evaluate the sum (4.11), note that, even though the operator algebra involving the ghost-matter mixing vertices is generally picture dependent (see the discussion below), all the $(N+4)$-point correlators entering the sum (4.11) have the same value for $s_1+t_1\neq 0,-1,-2$, $s_2\neq 0,-1,-2$,

$$\langle V_5^{(s_1-3)}(p_1)V_5^{(s_2-3)}(p_2)V_a^{(t_1-1)}(k_1)V_b^{(t_2-1)}(k_2)V_{a_1b_1}^{(r_1-1)}(q_1)...V_{a_Nb_N}^{(r_N-1)}(q_N)\rangle \equiv S_{N+4}(k_1,k_2,p_1,p_2,q_1...q_N). \tag{4.12}$$

This is because the full operator algebra for two V_5 operators has the form

$$\begin{aligned} V_5^{(s)}V_5^{(t)} &\sim \sum_{X,Y} C_X[X^{(s+t)}]+C_Y^{(s|t)}[Y^{(s+t)}]\,, \\ V_5^{(s)}[X^{(t)}] &\sim \sum_{Y} C_5^{(s|t)}[Y^{(s+t)}]\,, \end{aligned} \tag{4.13}$$

where $s,t\neq -2,-1,0$ and $[X]$ is the subclass of the picture independent operators (corresponding to the tower of standard perturbative massless and massive superstring states) and $[Y]$ is the tower of picture dependent (brane-like) vertices describing the non-perturbative sector of superstring fluctuations.

Note that the C_Y and C_5 coefficients are picture dependent while C_X are not. Then the second OPE of (4.13) ensures that the $[Y]$ part of the first OPE (4.13) with the picture dependent C_Y coefficients does not contribute to the correlators with two V_5 insertions, and the picture independence of the C_X coefficients for $s,t\neq -2,-1,0$ ensures the independence of the scattering amplitude (4.12) of the choice of s,t,r-pictures, ensuring its unitarity and factorization properties. Note that the operator product expansion of any operators from the $[X]$ subclass does not contain any operators from the $[Y]$ subclass (but the reverse is not true). This is why the brane-like states never appear as the intermediate poles in the perturbative amplitudes and do not affect the standard string perturbation theory. Using (4.12) we can now evaluate the sum (4.11) obtaining

$$A(p_1,p_2,k_1,k_2,q_1,...q_N)=\frac{1}{(2)!^2N!}\,S_{N+4}(p_1,p_2,k_1,k_2,q_1,...q_N)\,, \tag{4.14}$$

i.e. the presence of the $\Xi_\Gamma^{-1}(N+4;2)$ factor in the expression (4.11) ensures the correct normalization of the scattering amplitude and the corresponding term in the low-energy effective action. Next, substituting the OPE (4.10) into the amplitude (4.11) (multiplying the latter by the corresponding space-time fields) and performing the worldsheet integration we find that the OPE singularity leads to a logarithmic divergence in this correlator, and hence in the partition function, given by

$$\begin{aligned}
&A_\Lambda(p_1,p_2;k_1,k_2;q_1...q_N) = (ip_{1a})\log\Lambda\,\frac{\lambda^2 A_a A_b H_{a_1b_1}...H_{a_Nb_N}}{(2!)^2 N!\Xi_\Gamma(N+2;4)}\\
&\times\sum_{s,t,r}\frac{(N+6)!}{s_1!s_2!t_1!t_2!r_1!...r_{N-1}!(N+6-s_1-s_2-t_1-t_2-\sum r)!}\,C^{(s_1-3|s_1+t_1-4)}\\
&\times\langle V_5^{(s_1+t_1)}(p_1+k_1)V_5^{(s_2)}(p_2)V_b^{(t_2-1)}(k_2)V_{a_1b_1}^{(r_1-1)}(q_1)...\rangle\\
&+\{(s_1\leftrightarrow s_2;p_1\leftrightarrow p_2)+(t_1\leftrightarrow t_2;k_1\leftrightarrow k_2)+(s_1,t_1\leftrightarrow s_2,t_2;k_1,p_1\leftrightarrow k_2,p_2)\}.
\end{aligned}\tag{4.15}$$

Again, using (4.12) and substituting for the $C^{(s|t)}$ picture matrix we can perform the summation in (4.15) leading to

$$\begin{aligned}
A_\Lambda(p_1,p_2;k_1,k_2;q_1...q_N) &= (ip_{1a})\log\Lambda\,\frac{\lambda^2 A_a A_b H_{a_1b_1}...H_{a_Nb_N}}{2!N!}\\
&\times\big(1-\Xi_\Gamma^{-1}(N+4;2)F(N)\big)S_{N+3}(k_1+p_1,p_2,k_2,q_1...q_N)\,,
\end{aligned}\tag{4.16}$$

where

$$\begin{aligned}
&S_{N+3}(p_1,p_2,k,q_1...q_N)\\
&\equiv\langle V_5^{(s_1-3)}(p_1)V_5^{(s_2-3)}(p_2)V_b^{(t-1)}(k)V_{a_1b_1}^{(r_1-1)}(q_1)...V_{a_Nb_N}^{(r_N-1)}(q_N)\rangle\,,\\
&s_{1,2},t,r_1,...r_N\geq 0\,,\quad s_{1,2}\neq 1,2,3,\quad s_1+s_2+t+\sum_{j=1}^N r_j = N+5\,.
\end{aligned}\tag{4.17}$$

Ξ_Γ is defined by (4.5) and the function $F(N)$ is given by

$$\begin{aligned}
F(N) &= (N+2)^{N+6}\Bigg\{\frac{1}{2}\,\frac{(N+5)(N+6)}{(N+2)^2}+\frac{1}{6}\,\frac{(N+4)(N+5)(N+6)}{(N+2)^3}\\
&\times\left(1+\frac{1}{2}\frac{N+3}{N+2}\right)-\frac{(N+3)(N+4)(N+5)(N+6)}{(N+2)^4}\left(\frac{17}{72}\left(\frac{N+1}{N+2}\right)^{N+1}\right.\\
&\left.+\frac{1}{2}\left(\frac{N+1}{N+2}\right)^{N+2}\right)-\frac{1}{72}\,\frac{N(N+3)(N+4)(N+5)(N+6)}{(N+2)^5}\left(\frac{N+1}{N+2}\right)^N\\
&-\frac{1}{2}\,\frac{(N+4)(N+5)(N+6)}{(N+2)^3}\left(\frac{N+1}{N+2}\right)^{N+3}\Bigg\}.
\end{aligned}\tag{4.18}$$

As a result, the picture dependence of the OPE (4.10) leads to the dependence of the normalization of the logarithmic divergence (4.15) on the number of external insertions $V_{a_jb_j}$ (the operators other than V_5 and the photons). Obviously, in the absence of picture dependence (implying $\Xi_\Gamma(N+2;4)=(N+4)^{N+6}$, $C^{(s|t)}\equiv 1$ for any s,t and $F(N)\equiv 0$) there would be no dependence on N and this divergence could have been removed by the renormalization of the λ field in the lower order $(N+3)$-point correlation function proportional to

$$S_{N+3}(p_1,p_2;k_1;q_1,...,q_N)\sim A\lambda^2 H^n\langle V_5(p_1)V_5(p_2)V_a(k_1)V_{a_1b_1}(q_1)...V_{a_Nb_N}(q_N)\rangle. \tag{4.19}$$

The naive renormalization of the λ field removing the divergence would have been schematically given by (suppressing the momentum integration)

$$\lambda\to\lambda-C_aA_a\lambda\log\Lambda \tag{4.20}$$

(C_a are the structure constants), leading to the standard terms in the β function of the λ field. In the picture dependent case ($C^{(s|t)}=0$ for $s,t=0,-1,-2$) such a RG flow cannot cancel the divergence (4.16) in the partition function because the correlators (4.19) and (4.16) are differently normalized (the normalization of the latter depends on N and therefore the flow (4.20) cannot remove the divergence simultaneously for all the correlators). The resolution of this difficulty is that the divergence (4.16) is removed not by the standard quadratic renormalization of the λ field in the $(N+3)$-point correlator but by the cubic renormalization of the space-time fields inside the $(N+2)$-point function $\sim\lambda AH^n\langle V_5V_aV_{a_1b_1}...V_{a_Nb_N}\rangle$, i.e. in the lower order expansion term. Indeed, using the transversality condition for the photon, $k_aA^a(k)=0$, and momentum conservation we can recast the divergence (4.16) into the form

$$\begin{aligned}A_\Lambda(p_1,p_2;k_1,k_2;q_1...q_N)&=\frac{i}{N!}\Big(p_2^a+k_2^a+\sum_{j=1}^N q_j\Big)\lambda(p_1)\lambda(p_2)A_a(k_1)A_b(k_2)\\&\times\prod_{j=1}^N H_{a_jb_j}(q_j)\big(1-\Xi_\Gamma^{-1}(N+2;4)F(N)\big)\,S_{N+3}(k_1+p_1,p_2,k_2,q_1...q_N).\end{aligned} \tag{4.21}$$

Now consider the expansion term of order $N+2$ in the generating functional (4.1) which is given by

$$A_{N+2}(p,k,q_1...q_N)=\frac{1}{N!}\int_{p,k,q}\lambda(p)A_a(k)\prod_{j=1}^N H_{a_jb_j}(q_j)V_5(p)V^a(k)V^{a_jb_j}(q_j)\ , \tag{4.22}$$

and the operator-valued RG flows

$$\begin{aligned}
\lambda(p_1) &\to \lambda(p_1) + \log\Lambda \sum_{n=-3;n\neq-2,-1,0}^{\infty} i\alpha_n p_1^a \lambda(p_1) \int_{k,p} \lambda(p) A^a(k) \int_\Lambda V_5^{(n)}(k+p)\,,\\
A_b(k_1) &\to A_b(k_1) + \log\Lambda \sum_{n=-3;n\neq-2,-1,0}^{\infty} i\alpha_n k_1^a A_b(k_1) \int_{k,p} \lambda(p) A^a(k) \int_\Lambda V_5^{(n)}(k+p)\,,\\
H_{a_jb_j}(q_j) &\to H_{a_jb_j}(q_j) + \log\Lambda \sum_{n=-3;n\neq-2,-1,0}^{\infty} i\alpha_n q_j^a H_{a_jb_j}(k_1) \int_{k,p} \lambda(p) A^a(k) \int_\Lambda V_5^{(n)}(k+p)\,,
\end{aligned} \tag{4.23}$$

where, as previously, n labels the admissible pictures, $\int_\Lambda$ is the worldsheet integral cut off at the Λ scale, $\int_{k,p}$ stands for all the momentum integrals and α_n are some numbers yet to be determined. The transformation of (4.22) under the RG flows (4.23) gives

$$\begin{aligned}
&A_{N+2}(k_1,p_1,q_1...q_N) \to A_{N+2}(k_1,p_1,q_1...q_N)\\
&+ \frac{i}{N!} \log\Lambda \int_{k_2,p_2} \sum_{n=-3;n\neq-2,-1,0}^{\infty} \alpha_n\,(k_1^b + p_1^b + q_1^b + ...q_N^b)\\
&\times \lambda(p_1)\lambda(p_2)A_a(k_1)A_b(k_2) \prod_{j=1}^{N} H_{a_jb_j}(q_N)\\
&\times\langle|\rho_{(\Gamma,Z)}|^2 V_5^{(-3)}(p_1)V_5^{(n)}(k_2+p_2)V_a^{(-1)}(k_1)V_b^{(-1)}(k_2)V_{a_1b_1}^{(-1)}(q_1)...V_{a_Nb_N}^{(-1)}(q_N)\rangle\,.
\end{aligned} \tag{4.24}$$

Now it is clear that the sum (4.24) will be truncated at $n = N+3$ because the total ghost number of the correlator (4.24) must be equal to -2 and all the terms in the $\rho_{(\Gamma,Z)}$ factor have positive ghost numbers; in the meantime all the correlators with $n \leq N+3$ will contribute the factor $\sim S_{N+3}(p_1, p_2 + k_2, k_1, q_1...q_N)$. Thus the transformation of $A_{N+2}(k_1,p_1,q_1...q_N)$ under the flows (4.23) can be expressed as

$$\begin{aligned}
&A_{N+2}(k_1,p_1,q_1...q_N) \to A_{N+2}(k_1,p_1,q_1...q_N)\\
&+ \frac{i}{N!} \log\Lambda \int_{k_2,p_2} S_{N+3}(p_1,k_2+p_2,k_1,q_1..q_N) \sum_{n=-3;n\neq-2,-1,0}^{N+3} (k_1^b + p_1^b + q_1^b + ...q_N^b)\\
&\times \lambda(p_1)\lambda(p_2)A_a(k_1)A_b(k_2) \prod_{j=1}^{N} H_{a_jb_j}(q_N) \sum_{n=-3;n\neq-2,-1,0}^{N+3} \alpha_n\,.
\end{aligned} \tag{4.25}$$

Comparing this with (4.16) we conclude that the RG flow of A_{N+2} removes the divergence of (4.16) if we require

$$\alpha_{-3} + \sum_{n=1}^{N+3} \alpha_n = 1 - \Xi_\Gamma^{-1}(N+4;2)F(N)\,, \tag{4.26}$$

which leads to

$$\begin{aligned} &\alpha_{n+3} = \Xi_\Gamma^{-1}(n+4;2)F(n) - \Xi_\Gamma^{-1}(n+5;2)F(n+1)\,, \\ &\quad n = 0,1,2...\,, \\ &\alpha_{-3} + \alpha_1 + \alpha_2 = 1 - \Xi_\Gamma^{-1}(4;2)F(0). \end{aligned} \tag{4.27}$$

The choice (4.27) of the coefficients in the RG flows (4.23) ensures the unitarity of the string perturbation theory and the absence of UV divergences in the sigma-model (4.1). Curiously, the series in α_n converges to the inverse logarithm of the Feigenbaum's universality constant $\delta = 4.669...$ [16]

$$\alpha_{-3} + \sum_{n=1}^{\infty} \alpha_n \approx 0.649 \approx \frac{1}{\log\,\delta}.$$

Such a numerical coefficient (the inverse log of delta) (which, roughly speaking, "normalizes" the V_5 operator in the stochastic term) appears to be universal in all the calculations involving correlation functions with picture dependent vertices. In general, the β functions in the various ghost-matter mixing backgrounds (corresponding to the brane insertions) always contain the stochastic terms, universally normalized regardless of the details of the ghost-matter mixing. The underlying reasons for such a "Feigenbaum universality" in string theory are not clear at present; in principle, they should be related to the chaotization of the RG flows at certain critical values of λ (which is related to the viscosity and hence the Reynolds number of the "stringy liquid" – see the discussion below). Some preliminary and incomplete attempts to explain this phenomenon have been made in [17]. This appears to be an interesting and intriguing question to clarify.

Finally, transforming to position space and adding the kinetic terms, we find the RG flows (4.23) to be given by

$$\begin{aligned}
\frac{\partial\lambda}{\partial\tau}-(\vec{\nabla})^2\lambda+\lambda(\vec{A}\vec{\nabla})\lambda\int_\Lambda W_5(0)=0\,,\\
\frac{\partial\vec{A}}{\partial\tau}-(\vec{\nabla})^2\vec{A}+\lambda(\vec{A}\vec{\nabla})\vec{A}\int_\Lambda W_5(0)=0\,,\\
\frac{\partial H_{ab}}{\partial\tau}-(\vec{\nabla})^2H_{ab}+\lambda(\vec{A}\vec{\nabla})H_{ab}\int_\Lambda W_5(0)=0\,,\\
W_5(0)\equiv\sum_{n=-3;n\neq-2,-1,0}^{\infty}\alpha_nV_5^{(n)}(p=0)\,,
\end{aligned}\tag{4.28}$$

where $\tau\equiv\log\Lambda$. In the case when the λ field varies slowly, the RG equations (4.28) become the equations of fluid mechanics with viscosity $\nu\sim\frac{1}{\lambda}$ and velocity

$$\vec{v}=\vec{A}\int_\Lambda W_5.\tag{4.29}$$

The incompressibility of this liquid follows trivially from the transversality of the photon. The second equation of (4.28) is the Navier-Stokes equation for the velocity $\vec{v}$ operator. It does not yet contain the term with the gradient of pressure. In principle,the pressure term can be added as a result of the deterministic contribution of the OPE of two axions and gravitons to the photon's β function,

$$\lim_{z,w\to s}V_{B,G}(k;z,\bar{z})V_{B,G}(p,w,\bar{w})\sim|z-w|^{-2}(z+\bar{z}-w-\bar{w})^{-1}C^a(k,p)V_a(k+p)\,,\tag{4.30}$$

where

$$C_a\sim k_a+...\tag{4.31}$$

(here we skipped the terms cubic in the momentum). This OPE particularly gives rise to the term $\sim Tr(F(B+G)^2)$ in the expansion of the DBI effective action. Then the pressure can be expressed in terms of the axion and the graviton fields,

$$P\sim Tr(B+G)^2+\dots.\tag{4.32}$$

The random force term can also be obtained if one considers the higher order expansion terms in λ and V_5. In particular, the term cubic in λ contributes the stochastic term $\sim(\vec{\nabla}\lambda)^2\vec{\nabla}\lambda\int_\Lambda V_5$ to the photon's β function and can be interpreted as a random force acting in the liquid. Finally, consider the last equation of the RG flows (4.28). If the H field of (4.28) is a dilaton, i.e. $H_{ab}=$

$\varphi(\eta_{ab} - k_a\bar{k}_b - \bar{k}_a k_b)$, this equation is simply the well-known equation for the advection of the passive scalar by the turbulent liquid with the velocity $\vec{v}$ of (4.29) [3], [18], [19]. If H is a graviton, the equation describes the evolution of the stress-energy tensor of the fluid in the process of turbulence.

5. Discussion

In this paper we have shown that the renormalization group flows for the massless modes of the NSR superstring in the presence of D-branes have the form of the equations of fluid mechanics. The velocity distribution in the turbulent flow was shown to be related to the photon (D-brane's U(1) field) multiplied by the D-brane W_5 creation operator (4.29), (3.13). The dilaton has been interpreted as a passive scalar (e.g. a temperature) while the roles of the axion and the graviton in such a string-turbulence glossary have been related to the pressure and stress-energy tensor. It should be stressed that such a string-to-turbulence correspondence should be understood in terms of the off-shell renormalization group flows, rather than the on-shell scattering amplitudes. The results derived in this paper may suggest an isomorphism between the off-shell superstring theory in D-brane backgrounds and the degrees of freedom of fluid mechanics, in the manner of the well-known gauge-string correspondence [4], [2]. Just as confining QCD strings can be thought of being made of gluon fluxes confined to a thin tube, the strings of fluid mechanics can be related (using slightly simplistic language) to the vortices of a turbulent liquid. Thus it may be tempting to use the language and the formalism of the off-shell string theory to explore the mechanism of turbulence (as the straightforward analysis of the Navier-Stokes equation is known to be extremely difficult). In particular, one can hope to use the off-shell string theoretic formalism to derive Kolmogorov scaling or to study the statistical distribution of the vortices. In this paper we studied the evolution of the superstring massless modes in the non-equilibrium curved backgrounds rather than the stable ones created by static D-branes. These backgrounds appear when at a certain time one introduces a D-brane into the originally flat vacuum by making insertions of V_5 and the metric begins to fluctuate strongly. The hydrodynamical character of the evolution of the massless modes is an important property of string dynamics in such a fluctuating metric. Later on, however, the metric should stabilize to the usual D-brane-like backgrounds, as, in the limit of infinite stochastic time (corresponding to $\Lambda \to 0$) the stochastic process (4.23), (4.29) reaches the limit of thermodynamical equilibrium. This means that the usual D-brane geometry (e.g. the equilibrium configurations of the metric and the dilaton) should be the result of certain soliton-type solutions of the fluid mechanics equations. In particular, if we believe that our 4-dimensional Universe lives on the asymptotically flat boundary of AdS_5 (which is suggested by the standard gauge-string correspondence) one is led to the curious question whether the world we are living in has been formed as a result of a certain giant multi-dimensional turbulence in the early Universe. In other words, are we simply living on a hydrodynamical soliton formed somewhere in the bulk of a primordial fluid in higher dimensions?

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