# Physical applications of complex variables: problems and solutions

Rhonald C Lua

May 28, 2004

1. (Spiegel, problem 9-50) Referring to problem 10, (a) show that the speed of the fluid at any point $E$ [Fig. 9-14] on the surface of the cylinder is given by $2V_0|\sin\theta|$ and (b) determine at what points on the cylinder the speed is greatest.

   *Solution:*

   Here are some useful results from problem 10.

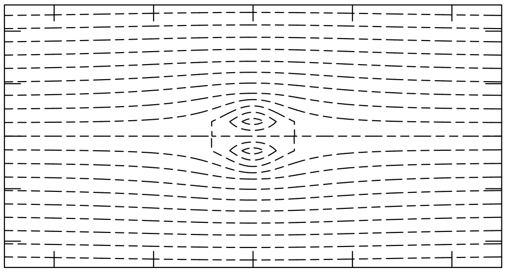

Figure 1: Fluid flow given by the complex potential $\Omega(z) = z + 1/z$ (using GNUplot). The lines depicted are lines of constant $\Psi = \Im\{\Omega(z)\}$ corresponding to streamlines.

The complex potential is given

$$\begin{aligned}\Omega(z) &= V_0\left(z + \frac{a^2}{z}\right) \\ &= \underbrace{\Phi(x,y)}_{velocity\,potential} + i \underbrace{\Psi(x,y)}_{stream function}\end{aligned}$$

(For analytic functions, each part automatically satisfies laplace's equation, as well as the Cauchy-Riemann equations.) Therefore the streamlines are given by

$$V_0\left(r-\frac{a^2}{r}\right)\sin\theta = V_0\left(y-\frac{a^2 y}{x^2+y^2}\right) = constant \tag{1}$$

The equipotential lines are given by

$$V_0\left(r+\frac{a^2}{r}\right)\cos\theta = constant \tag{2}$$

The circle $r = a$ represents a streamline ($r = 1$ in figure 1). It can be considered as a circular obstacle of radius $a$ placed in the path of the fluid.

The complex velocity is given

$$\overline{\frac{d\Omega}{dz}} = V_0\left(1-\frac{a^2}{r^2}\cos 2\theta\right) - i\frac{V_0 a^2}{r^2}\sin 2\theta \tag{3}$$

with magnitude

$$V = V_0\sqrt{1-\frac{2a^2\cos 2\theta}{r^2}+\frac{a^4}{r^4}} \tag{4}$$

Now on to the problem. At a point on the cylinder of radius $a$ centered at the origin, the speed is given by equation 4 after setting $r = a$

$$V(at\, surface) = V_0\sqrt{2-2\cos 2\theta} = 2V_0|\sin\theta| \tag{5}$$

The speed is clearly greatest at the top and bottom. (For an ellipse, the fluid speed would be given by $V_0(1 + b/a)$. Interpreted in the context of electrostatics, this result also provides a nice explanation for the existence of intense electric fields around sharp objects, e.g. $a \ll b$.)

*QED 05/09/04*

2. (Spiegel, problem 9-51) (a) If $P$ is the pressure at point $E$ of the obstacle in Fig. 9-14 of problem 10 and $P_\infty$ is the pressure far from the obstacle, show that

$$P - P_\infty = \frac{1}{2}\sigma V_0^2\left(1-4\sin^2\theta\right) \tag{6}$$

(b) Show that a vacuum is created at points $B$ (bottom) and $F$ (top) if the speed of the fluid is equal to or greater than $V_0 = \sqrt{2P_\infty/3\sigma}$. This is called *cavitation*.

*Solution:*

By Bernoulli's principle

$$\begin{aligned} P_\infty + \frac{1}{2}\sigma V_0^2 &= P + \frac{1}{2}\sigma V^2 \\ &= P + \frac{1}{2}\sigma V_0^2\left(4\sin^2\theta\right) \end{aligned}$$

Setting $P = 0$ and $\theta = \pm\pi/2$, one recovers $V_0 = \sqrt{2P_\infty/3\sigma}$.

*QED 05/09/04*

3. (Spiegel, 9-56) Show that the total moment on the cylindrical obstacle of Problem 10 is zero and explain physically.

   *Solution:*

   Might as well calculate the force also. The force components and the moment may be obtained from the Theorems of Blasius

$$\overline{F} = X - iY = \frac{1}{2} i\sigma \oint_C \left(\frac{d\Omega}{dz}\right)^2 dz \tag{7}$$

$$M = Re\left\{-\frac{1}{2}\sigma \oint_C z\left(\frac{d\Omega}{dz}\right)^2 dz\right\} \tag{8}$$

   Using $\frac{d\Omega}{dz} = V_0\left(1 - a^2/z^2\right)$

$$\overline{F} = \frac{1}{2} i\sigma \oint_C V_0^2\left(1 - 2\frac{2a^2}{z^2} + \frac{a^4}{z^4}\right) dz$$
$$M = Re\left\{-\frac{1}{2}\sigma \oint_C V_0^2\left(z - 2\frac{2a^2}{z} + \frac{a^4}{z^3}\right) dz\right\}$$

   From Cauchy's integral formula, the residue theorem, or the result

$$\oint_C \frac{dz}{(z-a)^p} = \begin{cases} 2\pi i & p = 1 \\ 0 & p = \text{integer} \neq 1 \end{cases} \tag{9}$$

   (which may be verified by switching to polar coordinates) both components of the force $X, Y$ vanishes. $M$ also vanishes because the integral is imaginary. The vanishing of the moment is also consistent with symmetry.

   *QED 05/10/04*

4. (Spiegel, 9-88) A circular cylinder obstacle of radius $a$ rests at the bottom of a channel of fluid which at distances far from the obstacle flows with velocity $V_0$

   (a) Prove that the complex potential is given by $\Omega(z) = \pi a V_0 \coth(\pi a/z)$

   (b) Show that the speed at the top of the cylinder is $\frac{1}{4}\pi^2 V_0$ and compare with that for a circular obstacle in the middle of a fluid.

   (c) Show that the difference in pressure between top and bottom points of the cylinder is $\sigma\pi^4 V_0^2/32$.

   *Solution:*

   From the conformal mapping table of Spiegel (p.207) the region in the $z - plane$ above the real-axis excluding the cylinder is mapped onto region above the real axis of the $w$-plane by the transformation

$$\Omega(z) = \pi a V_0 \coth(\pi a/z) \tag{10}$$

   ($w(u, v) = u + iv$ satisfy laplace's equation in the variables $u$ and $v$. The transformation by an analytic function to $w(x, y) = u(x, y) + iv(x, y)$ also satisfies laplace's equation as well as the boundary conditions. One can check that the cylindrical boundary is mapped onto the real line $[-1,1]$, points on the real axis map onto points on the real axis and points on the imaginary axis above the cylinder map onto points on the imaginary axis) Far from the obstacle, the complex potential becomes $\Omega(z) = \pi a V_0(z/\pi a) = V_0 z$.

The speed at the top of the cylinder is obtained by taking the derivative of the complex potential

$$\begin{aligned} \left.\frac{d\Omega}{dz}\right|_{z=i2a} &= \pi a V_0 \frac{-\pi a/(i2a)^2}{-\sinh^2(\pi a/(i2a))} \\ &= \frac{1}{4}\pi^2 V_0 \end{aligned} \tag{11}$$

By Bernoulli's principle $P_{\text{bottom}} - P_{\text{top}} = \frac{1}{2}\sigma V_{\text{t}op}^2 = \sigma \pi^4 V_0^2/32$

*QED 05/12/04*

The real and imaginary parts of the complex potential give the following velocity potential and stream function

$$\Phi(x,y) = \pi a V_0 \frac{\cosh\frac{\pi a x}{x^2+y^2}\sinh\frac{\pi a x}{x^2+y^2}\left(1-2\sin^2\frac{\pi a y}{x^2+y^2}\right)}{\left(\sinh\frac{\pi a x}{x^2+y^2}\cos\frac{\pi a y}{x^2+y^2}\right)^2+\left(\cosh\frac{\pi a x}{x^2+y^2}\sin\frac{\pi a y}{x^2+y^2}\right)^2} \tag{12}$$

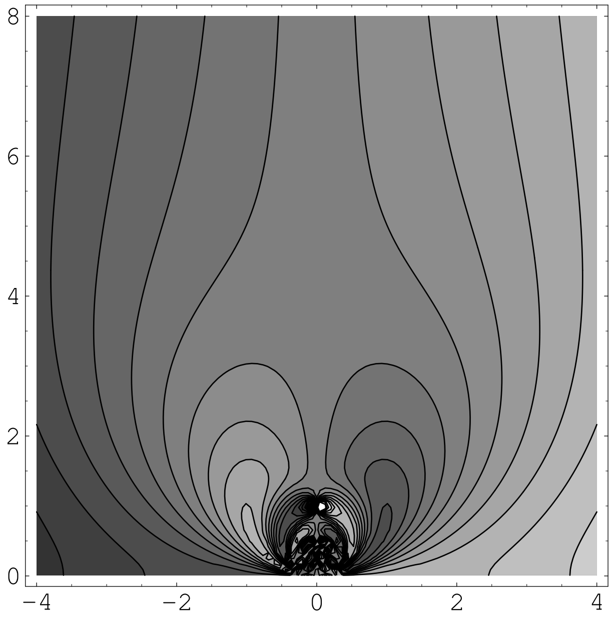

Figure 2: Velocity equipotentials for the complex potential $\Omega(z) = \pi a V_0 \coth(\pi a/z)$ with $a = 0$ (using Mathematica).

$$\Psi(x,y) = \pi a V_0 \frac{\cos\frac{\pi a y}{x^2+y^2}\sin\frac{\pi a y}{x^2+y^2}\left(1-2\cosh^2\frac{\pi a x}{x^2+y^2}\right)}{\left(\sinh\frac{\pi a x}{x^2+y^2}\cos\frac{\pi a y}{x^2+y^2}\right)^2+\left(\cosh\frac{\pi a x}{x^2+y^2}\sin\frac{\pi a y}{x^2+y^2}\right)^2} \quad (13)$$

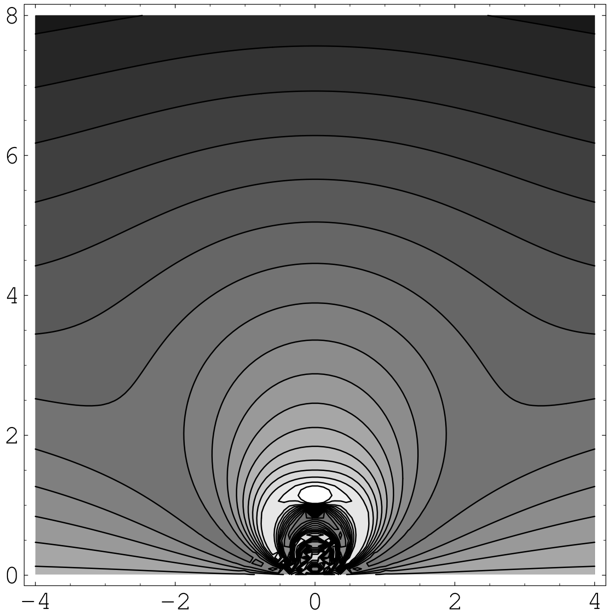

Figure 3: Streamlines for the complex potential $\Omega(z) = \pi a V_0 \coth(\pi a/z)$ with $a = 0$ (using Mathematica).

5. (Spiegel, 9-61) Fluid flows between the two branches of the hyperbola $ax^2 - by^2 = 1$, $a > 0$, $b > 0$. Prove that the complex potential for the flow is given by $K\cosh^{-1}\alpha z$ where $K$ is a positive constant and $\alpha = \sqrt{ab/(a+b)}$.

   *Solution:*

   Under the transformation, $w = f(z) = K\cosh^{-1}\alpha z$, ‘surfaces’ of constant-$v$, $v = C$, satisfy

$$\begin{aligned} x &= \frac{1}{\alpha}\cosh\frac{u}{K}\cos\frac{C}{K} \\ y &= \frac{1}{\alpha}\sinh\frac{u}{K}\sin\frac{C}{K} \end{aligned}$$

$$\alpha^2\frac{x^2}{\cos^2\frac{C}{K}} - \alpha^2\frac{y^2}{\sin^2\frac{C}{K}} = 1 \quad (14)$$

   If we identify

$$\begin{aligned} a &= \frac{\alpha^2}{\cos^2\frac{C}{K}} \\ b &= \frac{\alpha^2}{\sin^2\frac{C}{K}} \end{aligned}$$

Then

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{\alpha^2} \tag{15}$$

*QED 05/20/04*

6. (Spiegel, 9-86) Two infinitely long cylindrical conductors having cross-sections which are confocal ellipses with foci at $(-c,0)$ and $(c,0)$ (major axis lengths of $R_1$,$R_2$) are charged to constant potentials $\Phi_1$ and $\Phi_2$ respectively. Show that the capacitance per unit length is equal to

$$\frac{2\pi}{\cosh^{-1}(R_2/c)-\cosh^{-1}(R_1/c)} \tag{16}$$

[*Hint.* Use the transformation $z=c\cosh w$.]

*Solution:* We shall transform the coaxial elliptic cylinders into a system with a simpler geometry, then calculate the capacitance. This capacitance is identical to that of the original system, as I shall try to demonstrate at the end.

By using the transformation, $z=c\cosh w$, an ellipse in the $z$-plane is mapped onto a vertical line segment in the $w$-plane. In terms of $u$, the major and minor axes of the ellipse is given by $a=c\cosh u_1$ and $b=c\sinh u_2$. After mapping the two coaxial cylinders onto two lines, the distance between the lines (or plates) is

$$d=u_2-u_1=\cosh^{-1}(R_2/c)-\cosh^{-1}(R_1/c) \tag{17}$$

The capacitance of a parallel plate capacitor (per unit length) is given by

$$\begin{aligned} C &= \epsilon_0\frac{\text{(Area per unit length; range of v for one cycle of ellipse)}}{\text{distance between plates}} \\ &= \epsilon_0\frac{2\pi}{\cosh^{-1}(R_2/c)-\cosh^{-1}(R_1/c)} \end{aligned}$$

This result is in S.I. units, which is not quite the same as that indicated in the problem after setting $\epsilon_0=1/4\pi$.

Another way to get this result is to use a slightly different transformation, one which maps an ellipse onto a circle. This transformation can be broken into two steps.

- Ellipse to line segment: $w_1=\cosh^{-1}(z/c)$.
- Line segment to circle: $w_2=e^{w_1}$.

The relationship between the radius of the circle in the $w_2$-plane and the major axis of the ellipse in the $z$-plane is given by

$$R_{w_2}=a/c+\sqrt{(a/c)^2-1} \tag{18}$$

The capacitance of two coaxial circular cylinders is given by

$$C = \frac{2\pi\epsilon_0}{\ln(\text{larger radius})-\ln(\text{smaller radius})}$$

$$\begin{aligned}
&= \frac{2\pi\epsilon_0}{\ln(a_2+b_2)-\ln(a_1+b_1)} \\
&= \frac{2\pi\epsilon_0}{\ln(R_2/c+\sqrt{R_2^2-c^2}/c)-\ln(R_1/c+\sqrt{R_1^2-c^2}/c)} \\
&= \frac{2\pi\epsilon_0}{\cosh^{-1}(R_2/c)-\cosh^{-1}(R_1/c)}
\end{aligned}$$

(BTW, if the smaller ellipse degenerates into a line (or plate), the capacitance becomes $\frac{2\pi\epsilon_0}{\cosh^{-1}(R_2/R_1)}$.)

It was assumed at the beginning that the capacitance is invariant under the transformations. This may be seen thru the invariance of the following integral representing the surface integral of the electric field (i.e. charge) under a change of variable,

$$\begin{aligned}
\oint -\frac{d\Omega}{dz}dz &= \oint -\frac{d\Omega}{dw}\frac{dw}{dz}dz \\
&= \oint E_x dx + E_y dy + i\oint E_x dy - E_y dx \\
&= \oint E_t ds + i\oint E_n ds \\
&= 0 + i(\text{surface integral of normal component of } \vec{E})
\end{aligned}$$

*05/24/04*

7. (Paterson XVII.2) Flow along a plane with a semicircular bump on it, sketched in fig.XVII.52, has potential $w(z) = U(z+a^2/z)$. Use the transformation $z \to z^{1/2}$ to solve for the flow in a corner which has a bump in it.

   *Solution:* The streamfunction is similar to that in problem 1, with $r$ squared and the angle doubled.

$$\Psi(x,y) = U\left(r^2 - \frac{a^2}{r^2}\right)\sin 2\theta = 2Uxy\left(1-\frac{a^2}{r^4}\right) \tag{19}$$

*05/27/04*

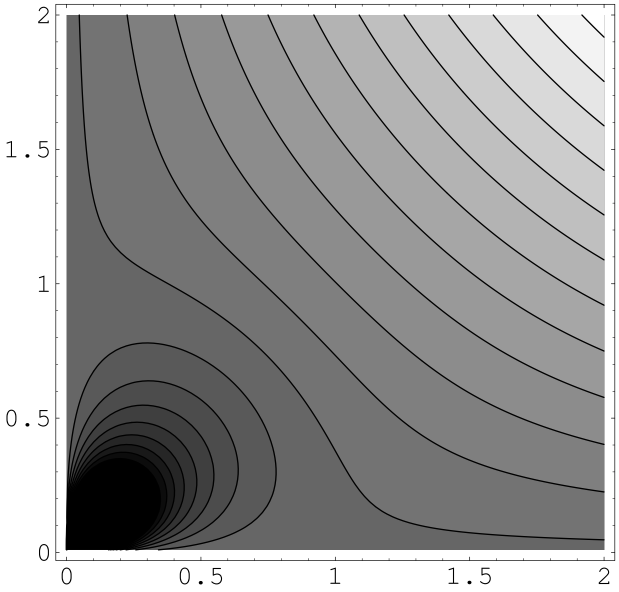

Figure 4: Streamlines for flow in a corner with a bump (using Mathematica).